Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 12

Mathematics Assignment 01

The Quadratic Function

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This is the first Mathematics assignment from COMPOS Y12. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in webinars, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

1 The Quadratic Function

The function in the form $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$ is known as the quadratic function. The graph of $y = f(x)$ is a *parabola*.

Please watch the videos on parabolas on this Khan Academy page. Some further information about quadratic functions and parabolas can be found on the AMSI website.

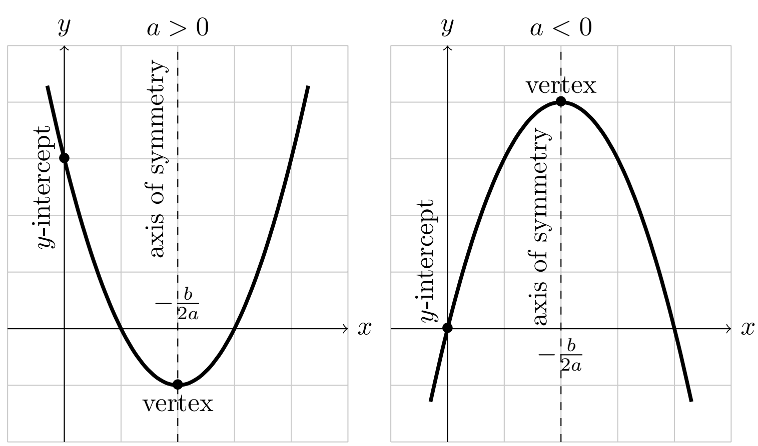

Figure 1: Salient features of a parabola.

Problem 1 (3 marks). Consider the graph $y = x^2 - 4x + 1$

a) By completing the square[1] show that the coordinates of the minimum point of the parabola are $(2, -3)$.

b) Show that $x = 2$ is the line of symmetry of the parabola $y = f(x) = x^2 - 4x + 1$, i.e. $f(2-x) = f(2+x)$ for any value of x.

c) Show that the the coordinates of the y-intercept of the parabola — i.e., the point where it crosses the y-axis — are $(0, 1)$.

Problem 2 (8 marks). Consider the graph $y = ax^2 + bx + c$.

a) The *vertex* of a parabola is its extreme point: minimum for $a > 0$ and maximum for $a < 0$. By completing the square or otherwise show that the coordinates of the vertex are $\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$.

b) Show that $x = -\frac{b}{2a}$ is the line of symmetry of the parabola $y = f(x) = ax^2 + bx + c$, i.e. $f\left(x - \frac{b}{2a}\right) = f\left(-\frac{b}{2a} - x\right)$ for any value of x.

c) Show that the the coordinates of the y-intercept of the parabola are $(0, c)$.

Problem 3 (2 marks). Find the parameters a, b, c of the two parabolas in Fig. 1. Assume that the grid step in the figure is 1.

Problem 4 (3 marks). Find the parameters a, b, c of the function $f(x) = ax^2 + bx + c$ if the graph $y = f(x)$ passes through the point $B(4, 5)$ and has a vertex at $A(2, -3)$.

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[1] *Completing the square* means writing

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + c.$$

This is a common technique, used e.g. in solving quadratic equations, see Sec. 2.

Regular or decimal fractions?

Suppose you are given the equation $24x = 6$. Should your answer be $x = 1/4$ or $x = 0.25$?

While the two numbers are formally identical, a decimal fraction traditionally implies limited precision — that you don't know the answer beyond the significant figures given. In maths problems, this is usually not the case: the numbers are exact. To indicate this, you should use a regular fraction, writing $x = 1/4$.

In physics, in contrast, numbers are not known precisely. Consider the question: *A tortoise walked 6 meters in 24 seconds. What has been its speed?* Even though this is seemingly the same problem, you should answer "0.25 m/s", to implicitly indicate the precision, with which the answer is known. In theoretical equations and symbolic answers, however, you should still use regular fractions. For example, you should write $s = \frac{1}{2}at^2$ rather than $s = 0.5at^2$ — because $\frac{1}{2}$ here is an exact value.

Generally you should avoid mixing regular and decimal fractions. Answers should not contain decimals under a square root.

2 Quadratic equations and Vieta's formulae

A general quadratic equation can be written in the form

$$ax^2 + bx + c = 0, \tag{1}$$

where $a \neq 0$. It can be solved e.g. by completing the square, giving rise to the well-known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

Read about this method on Isaac Physics and review the example therein.

Problem 5 (2 marks). Solve the following quadratic equations by completing the square. Do not use the formula (2).

a) $x^2 + 3x - 40 = 0$ b) $3x^2 + 5x - 2 = 0$

Example 1. Solve by substitution: $(2x - 1)^4 - (2x - 1)^2 - 12 = 0$.

Solution: Substituting $u = (2x-1)^2$ makes the equation $u^2 - u - 12 = 0$. This can be solved using formula (2): $u_1 = \frac{1+\sqrt{(-1)^2-4\times(-12)}}{2\times 1} = \frac{1+7}{2} = 4$ and $u_2 = \frac{1-\sqrt{(-1)^2-4\times(-12)}}{2\times 1} = \frac{1-7}{2} = -3$.

We now have 2 equations:

$(2x - 1)^2 = 4$ and $(2x - 1)^2 = -3$.

The second equation does not have any solutions, as the square of a number is always positive, so it cannot be equal to -3. The other equation is fairly easy to solve:

$(2x-1)^2 = 4 \Rightarrow 2x-1 = \pm 2 \Rightarrow x_1 = \frac{3}{2};\ x_2 = -\frac{1}{2}.$

Problem 6 (3 marks). Solve the following equations by substitution. Give answers in exact form:

a) (PAT[2] 2009) $x^4 - 13x^2 + 36 = 0$ **b)** $(x+2)^4 + 2x^2 + 8x - 16 = 0$ **c)** $4x^4 - (b+36)x^2 + 9b = 0$

Problem 7 (2 marks). By using the formula (2) for the roots of a quadratic equation, show that the two roots of a quadratic equation obey the following relations:

$$x_1 x_2 = \frac{c}{a}; \qquad x_1 + x_2 = -\frac{b}{a}. \tag{3}$$

These results are known as *Vieta's Formulae.*

Example 2. Write down a quadratic equation with the roots -3 and 5.

Solution: Let us set $a = 1$ for simplicity (*why* are we allowed to do this?). Then Vieta's formulae take the form $c = x_1 x_2 = -3 \times 5 = 15$ and $-b = x_1 + x_2 = -3 + 5 = 2$.

Answer: $x^2 - 2x - 15 = 0$

Problem 8 (3 marks). Solve the following quadratic equations using Vieta's formulae[3]. Write your answers down and check using a calculator and/or the discriminant formula:

a) $x^2 - 8x + 12 = 0$ b) $x^2 - 118x + 777 = 0$ c) $769x^2 - 175x - 594 = 0$

Example 3. Without finding the roots of the quadratic equation $2x^2 - 5x - 4 = 0$ explicitly, find:
a) $x_1^2 + x_2^2$; b) $\frac{1}{x_1^2} + \frac{1}{x_2^2}$.

Solution:

Using Vieta's formulas, we write: $x_1 x_2 = \frac{c}{a} = -\frac{4}{2} = -2;$ $\qquad x_1 + x_2 = -\frac{b}{a} = -\frac{-5}{2} = \frac{5}{2}.$

a) Using $(x_1 + x_2)^2 = x_1^2 + 2x_1 x_2 + x_2^2$ and subtracting $2x_1 x_2$ from both sides, we find

$$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = \left(\frac{5}{2}\right)^2 - 2(-2) = \frac{25}{4} + 4 = \frac{41}{4}.$$

b) Using the previous result, we have:

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} = \frac{x_2^2 + x_1^2}{x_1^2 x_2^2} = \frac{41/4}{(-2)^2} = \frac{41}{16.}$$

[2] Oxford Physics undergraduate admissions test.
[3] Using Vieta's formulae is sometimes referred to as *solving quadratic equations in your head*

Problem 9 (8 marks). Without finding the roots of the quadratic equation $2x^2 - 7x - 3 = 0$, find:

a) $x_1^2 + x_2^2$ b) $x_1 x_2^3 + x_2 x_1^3$ c) $\frac{x_1}{x_2^2} + \frac{x_2}{x_1^2}$ d) $x_1^4 + x_2^4$

Find the roots x_1 and x_2 and confirm (using a calculator) your answers to a)–d).

3 Factorisation

As can be seen by inspection, the equation of the form

$$(x - x_1) \times (x - x_2) \times \ldots \times (x - x_n) = 0$$

has (only) the roots $x_1, x_2, \ldots, x_n$. This fact is often handy in solving problems.

Example 4 (PAT 2010). Show that $x = -1$ is a root of the polynomial equation $x^3 + 2x^2 - 5x - 6 = 0$, and find the other two roots.

Solution: We see by substitution that $x = -1$ is indeed a root. This means that $(x + 1)$ can be factored out of the polynomial in the left-hand side of the equation at hand. We have

$$x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6).$$

Solving the quadratic equation $x^2 + x - 6 = 0$, we find the remaining two roots $x = -3$ and $x = 2$.

How to divide polynomials: Khan Academy video.

Problem 10 (4 marks). Factorise the following expressions fully:

a) $3b^2 + 8b + 5$ b) $20q^2 + 31q + 12$ c) $-36h^4 + 25h^2 - 4$ d) $-8x^2 - 10xy - 3y^2$

Problem 11 (modified PAT 2011, 4 marks). Given that $x^2 + x - 2$ is a factor of $3x^4 + 23x^3 + 7x^2 - 47x + 14$, find all the roots of this polynomial.

4 Quadratic equations with parameters

In Eq. (2), the quantity $D = b^2 - 4ac$ is known as the *discriminant*. It determines the number of roots of the equation:

- if $D > 0$, the quadratic equation has two real distinct roots $x_1 = \dfrac{-b + \sqrt{D}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{D}}{2a}$.
- if $D = 0$, the quadratic equation has one real distinct root $x_1 = \dfrac{-b}{2a}$.
- if $D < 0$, the quadratic equation has no real roots.

There are many problems, in which you are asked to analyze how many roots a differential equation has dependent on a certain *parameter*. This is illustrated by the following example.

Example 5 (PAT 2015). For what values of m does $4x^2 + 8x - 8 = m(4x - 3)$ have no real solutions?

Solution: Let's rewrite the equation in the standard form

$$4x^2 + (8 - 4m)x + (3m - 8) = 0$$

and find the discriminant:

$$D = (8 - 4m)^2 - 16(3m - 8) = 16m^2 - 112m + 196 = 16(m^2 - 7m + 12)$$

(the last step is not necessary, but simplifies further calculations). The original equation has no real roots when this discriminant is negative. Solving the equation $D = 0$, we find $m_1 = 4$ and $m_2 = 3$. D is negative when m is between these two values (why?).

Answer: $3 < m < 4$.

Problem 12 (4 marks). Solve the following quadratic equations for all values of b. Give answers in exact form in terms of b and identify the ranges of b for which the equations have zero, one and two roots.

a) $x^2 - 3bx + 2b^2 = 0$ b) $(b + 1)x^2 - 2x + 1 - b = 0$

Problem 13 (4 marks). For which values of a does the equation

$$(a + 2)x^2 + 2(a + 2)x + 2 = 0$$

have exactly one real root?

Problem 14 (4 marks). For which values of parameter k does is the vertex of the parabola $y = x^2+2kx+13$ 5 units away from the origin?

Problem 15* (4 marks). For which values of a does the equation

$$(a^2 + 5a + 6)x^2 + (a^2 - 9)x + 57 - 32a - 17a^2 = 0$$

have more than two distinct real roots?

5 Equations with rational expressions

The equations in this section appear complicated, but can be reduced to quadratic by transformation. To familiarize yourself with a couple of examples, watch the Khan Academy Video 1 and Video 2.

An important caveat in solving equations with rational expressions is to check if the denominator(s) turn to zero for the values of x you have found. If this is the case, you must exclude those roots from your answer.

Problem 16 (6 marks). Solve the following equations. Give answers in exact form:

a) $\dfrac{x^2-2x+1}{x-3}+\dfrac{x+1}{3-x}=4$ b) $\dfrac{4-3x}{x+1}+\dfrac{x+1}{4-3x}=\dfrac{50}{7}$ c)* $\dfrac{25}{4x^2+1}-\dfrac{8x+29}{16x^4-1}=\dfrac{18x+5}{8x^3+4x^2+2x+1}$

Problem 17 (4 marks). Solve the following equation for all values of a. Give answers in exact form in terms of a and investigate how many roots the equation has dependent on a

$$\frac{x^2-(a+1)x+2a-2}{3x^2+3x-5}=0.$$

Problem 18 (4 marks). A Magnesium-Aluminium alloy contains 22kg of Aluminium. 15kg of Magnesium was added. As a result, the mass concentration of Magnesium in the alloy increased by 33 percentage points. What was the initial total mass of the alloy?