Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 10

Mathematics Assignment 01

The Quadratic Function

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This is the first Mathematics assignment from COMPOS for Y10. The assignment goes into the topic in more detail than you have done in school. There are links to online videos which we encourage you to watch. You are free to do your own reading around this topic also.

This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. Some of the problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Harder problems are labelled * or **.

Very similar problems will be discussed in the webinars so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

To successfully complete this assignment you will need to be familiar with the following formulae:

- Difference of two squares formula:
$$a^2 - b^2 = (a-b)(a+b) \tag{1}$$
- Square of a sum formula:
$$(a+b)^2 = a^2 + 2ab + b^2 \tag{2}$$
- Square of a difference formula:
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{3}$$

An explanation can be found in this Khan Academy web page.

Total 65 marks.

1 Solving quadratic equations

The goal of this assignment is to introduce the concepts of *function* and *quadratic function*. To understand these, we will first consider quadratic equations. We will start with intuitive equations that can be solved by inspection.

1.1 Solving by inspection

Example 1. Solve $x^2 = 9$.

Solution: You probably already found a solution: $x = 3$, however to solve an equation means to find all solutions or show that there are none. So after a more careful consideration you will find that this equation is also true when $x = -3$. Answer: $x = \pm 3$.

Solutions to equations are called *roots*. For example, the roots of the equation $x^2 = 9$ are ± 3.

In the above example we could have a different expression instead of x.

Example 2. Solve $(2x - 7)^2 = 9$.

Solution: Here we have $(2x-7)$ instead of x. You can read it as "something squared is 9". This "something" must be 3 or -3. So now instead of $x = \pm 3$ we have $2x - 7 = \pm 3$, which is actually just 2 linear equations: $2x - 7 = 3$ and $2x - 7 = -3$ which are easy to solve. Answer: $x_1 = 5$; $x_2 = 2$.

Problem 1 (5 marks). Solve equations:

a) $x^2 = 36$; b) $x^2 = 13$; c) $(x - 3)^2 = 9$;

d) $(x + 2)^2 = 11$; e)* $(4x - 7)^2 = (5x + 13)^2$.

Another intuitive example of a quadratic equation is

Example 3. Solve $x^2 = 5x$.

Solution: We are tempted to divide both sides of the equation by x, which leads us to a root $x = 5$. But by doing so, we must be careful not to divide by zero. Indeed, we can see that $x = 0$ is also a valid root, and "lost" it by dividing by x thoughtlessly. Answer: $x_1 = 5$; $x_2 = 0$.

When dividing both sides of an equation by an expression, always check if this expression can be zero.

As before, in the above example we could have a different expression instead of x.

Example 4. Solve $(3x + 2)^2 = 5(3x + 2)$.

Solution: Here we have $(3x + 2)$ instead of x. You can read it as "something squared is 5 times that something". From the previous example we know that this "something" must be either 0 or 5. So now we have $3x + 2 = 0$ and $3x + 2 = 5$. Solving these two linear equations we get the Answer: $x_1 = -\frac{2}{3}$; $x_2 = 1$.

Problem 2 (8 marks). Solve equations:

a) $x^2 = 2x$; b) $(x+7)^2 = x+7$; c) $(x-8)^2 = 17(x-8)$;

d) $7 - 5x = (5x-7)^2$; e) $7(x-2)^2 = 6x - 12$;

f) $(9-4x)(x+5) = 5(x+5)$; g) $(5-3x)(7x-4) = (7x-4)^2$;

h)* $(11x-2)^2 - (7x+4)^2 = 0$.

Problem 3* (3 marks).

a) Solve the equation $x^2 + y^2 = 0$;

b) Hence, solve the equation $(x^2-9)^2 + (6x-2x^2)^2 = 0$.

1.2 Solving by factorisation

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0, \tag{4}$$

where a, b, c are real numbers.[1] We shall assume that $a \neq 0$ — otherwise such an equation would not be quadratic. The left-hand side of a quadratic equation — the expression $ax^2 + bx + c$ — is called a *quadratic polynomial.*

Let us first assume that $a = 1$, so the expression takes the form $x^2 + bx + c$. A useful technique of that helps *guessing* the solution of a quadratic equation is *factorising* its left-hand side — that is, presenting it as a product of *binomials*

$$x^2 + bx + c = (x+p)(x+q). \tag{5}$$

Example 5. Factorise $x^2 - 13x + 30$.

Solution: As discussed in the video, we need to find two numbers that add to -13 and multiply to 30. The numbers are -3 and -10. Answer: $x^2 - 13x + 30 = (x-3)(x-10)$.

This is how you were (most likely) taught in school, however the values of p and q have very specific meanings — they are opposite to the roots of the equation, i.e. $p = -x_1$ and $q = -x_2$. So from now on we will be writing the factorisation in the form

$$x^2 + bx + c = (x-x_1)(x-x_2), \tag{6}$$

[1] This is often written as $a, b, c \in \mathbb{R}$. $\mathbb{R}$ is the set of real numbers, you can watch this Nerdstudy video for more information.

where x_1 and x_2 are the roots of the polynomial: if x equals either one of these numbers, the expression (6) becomes zero.

Suppose, for example, that we are given the quadratic equation $x^2 - 3x + 2 = 0$. Because we can write $x^2 - 3x + 2 = (x-1)(x-2)$, the roots are $x_1 = 1$ and $x_2 = 2$.

But how can we factorise a quadratic polynomial? Let us make a transformation

$$(x - x_1)(x - x_2) = x^2 - (x_1 + x_2)x + x_1x_2. \tag{7}$$

Comparing this with Equation (6), we see that

$$x_1x_2 = c; \qquad x_1 + x_2 = -b \tag{8}$$

(the above equalities are called *Vieta's formulae*). That is, we need to guess two numbers that add up to $-b$ and whose product is c.

In the above example, we have $x_1x_2 = 2 = c$ and $x_1 + x_2 = 3 = -b$.

Example 6. Factorise $x^2 - 3x - 88$ using Vieta's formulae.

Solution: We need to find two *roots* x_1 and x_2 that add to 3 and multiply to -88:

$$\begin{aligned} x_1 \times x_2 &= -88 \\ x_1 + x_2 &= 3 \end{aligned}$$

The roots are $x_1 = -8$ and $x_2 = 11$.

Using formula (6) we get

Answer: $x^2 - 3x - 88 = (x+8)(x-11)$.

Example 7. Solve using Vieta's formulae: $x^2 + 9x + 14 = 0$.

Solution: We need to find two roots that add to -9 and multiply to 14:

$$\begin{aligned} x_1 \times x_2 &= 14 \\ x_1 + x_2 &= -9 \end{aligned}$$

These are $x_1 = -2$ and $x_2 = -7$.

Answer: -2; -7.

Problem 4 (4 marks).

a) Find the roots, and hence factorise the expressions:

i) $x^2 - 8x + 15$; ii) $x^2 - 8x - 33$; iii) $x^2 - 102x - 999$.

b) Write down the solutions to each equation:

i) $x^2 - 8x + 15 = 0$; ii) $x^2 - 8x - 33 = 0$; iii) $x^2 - 102x - 999 = 0$.

Sanity check

Whenever you solve a problem, you should think simple tests to see if your result makes sense. With equations, this is easy: you substitute the roots you found into the equation and check if the equality holds. In example (7) you have $x_1 = -2$ and $x_2 = -7$, substituting these into the original equation:

$x_1^2 + 9x_1 + 14 = (-2)^2 + 9 \times (-2) + 14 = 4 - 18 + 14 = 0$;

$x_2^2 + 9x_2 + 14 = (-7)^2 + 9 \times (-7) + 14 = 49 - 63 + 14 = 0$.

You are not required to submit the sanity check as a part of your assignment, but you should *always* do it on your own. Otherwise you may end up submitting a solution with a silly mistake, which could easily be avoided.

Express sanity check

Doing a full sanity check as above can be time consuming, so you can use a quicker test: $x_1 \times x_2 = c$. So in example (7): $x_1 \times x_2 = (-2) \times (-7) = 14$ which is the same as the value of c[1].

[1] Note that if $a \neq 1$ the product becomes $x_1 \times x_2 = \dfrac{c}{a}$.

Example 8. Write down a quadratic equation with roots -3 and 5.

Solution. Vieta's formulae take the form $c = x_1 x_2 = -3 \times 5 = 15$ and $-b = x_1 + x_2 = -3 + 5 = 2$.

Answer: $x^2 - 2x - 15 = 0$

Problem 5 (3 marks). Write down a quadratic equation with roots:

a) 5 and -2; b) -3 and 3; c)* $2 - \sqrt{5}$ and $2 + \sqrt{5}$

Importantly, not every quadratic polynomial can be factorised — that is, not every quadratic equation has solutions. For example, the equation $x^2 + 1 = 0$ has no roots. In the next section we will learn how to find out how many roots a given quadratic equation has.

Problem 6 (2 marks). Find b and x_2, if $x^2 + bx - 10 = 0$ and $x_1 = 5$.

Example 9.

a) Factorise $x^4 - 5x^2 + 4$;

b) solve $x^4 - 5x^2 + 4 = 0$.

Solution:

a) Note that if we substitute $u = x^2$, the expression becomes $u^2 - 5u + 4$, which can be factorised into $u^2 - 5u + 4 = (u-1)(u-4)$. Substituting x^2 back into the expression and applying the difference of two squares formula: $x^4 - 5x^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$.

b) $(x-1)(x+1)(x-2)(x+2) = 0$. A product is zero if one of the factors is zero. Hence, we have

$(x-1) = 0$ or $(x+1) = 0$ or $(x-2) = 0$ or $(x+2) = 0$

Which gives the following roots:

$x_1 = 1$, $x_2 = -1$, $x_3 = 2$, $x_4 = -2$.

Problem 7 (4 marks). Solve the equations:

a) (PAT[2] 2009) $x^4 - 13x^2 + 36 = 0$; b) $x^4 - 9x^2 - 112$=0; c)* $(x+5)^4 - 9(x+5)^2 + 18 = 0$.

More generally, as can be seen by inspection, the equation of the form

$$(x - x_1) \times (x - x_2) \times \ldots \times (x - x_n) = 0$$

has (only) the roots $x_1, x_2, \ldots, x_n$. This fact is often handy in solving problems.

What to do if we have a quadratic equation with $a \neq 1$? the simplest approach is to rewrite the equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{9}$$

Example 10. Find the roots of the equation $2x^2 - 3x - 2 = 0$ and hence factorise its left-hand side.

Solution: We divide both sides by 2, obtaining $x^2 - \frac{3}{2}x - 1 = 0$, and use Vieta's formulae to guess the roots:

$$\begin{aligned} x_1 \times x_2 &= -1 \\ x_1 + x_2 &= \tfrac{3}{2} \end{aligned}$$

the roots are $x_1 = 2, x_2 = -\frac{1}{2}$. Hence we have

$x^2 - \frac{3}{2}x - 1 = (x-2)\left(x + \frac{1}{2}\right)$.

[2] Oxford Physics undergraduate admissions test.

However, the polynomial we need to factorise is twice the above polynomial. We can write

$$2x^2 - 3x - 2 = 2(x^2 - \frac{3}{2}x - 1) = 2(x-2)\left(x + \frac{1}{2}\right) = (x-2)(2x+1).$$

Of course, we could also write the answer as $(2x-4)\left(x + \frac{1}{2}\right)$ or leave it as $2(x-2)\left(x + \frac{1}{2}\right)$.

Problem 8 (1 marks). Solve the equation $7x^2 + 22x + 15 = 0$ and factorise its left-hand side.

You can see a neat trick to guess the roots of a quadratic expression with $a \neq 1$ in this Mr Chernov's video.

Problem 9 (2 marks). Generalise Vieta's formulae (8): What are the sum and the product of the two roots of a quadratic equation (4) equal to if $a \neq 1$?

More uses of Vieta's formulae

Sometimes it is easy to guess one of the roots of the equation, then the second root can be found simply by solving $x_1 \times x_2 = \dfrac{c}{a}$, as demonstrated in the following example.

Example 11. Solve $2023x^2 - 1000x - 1023 = 0$.

Solution: The equation clearly has a root $x_1 = 1$. The second can be found by solving $x_1 \times x_2 = \dfrac{c}{a} \Rightarrow 1 \times x_2 = \dfrac{-1023}{2023} \Rightarrow x_2 = -\dfrac{1023}{2023}$.

Special case of Vieta's

It follows from Vieta's formulae and demonstrated in the above example, that in a quadratic equation $ax^2 + bx + c = 0$,

- if $a + b + c = 0$, then $x_1 = 1$, $x_2 = \frac{c}{a}$;
- if $a - b + c = 0$, then $x_1 = -1$, $x_2 = -\frac{c}{a}$.

Example 12. Solve:

a) $x^2 - 7x + 6 = 0$; b)$13x^2 + 25x + 12 = 0$;

Solution:

a) Looking at the coefficients we notice that $a + b + c = 1 + (-7) + 6 = 0$,

so $x_1 = 1$, $x_2 = \dfrac{c}{a} = \dfrac{6}{1} = 6$.

b) Similar to a), we notice that $a - b + c = 13 - 25 + 12 = 0$,

so $x_1 = -1$, $x_2 = -\dfrac{c}{a} = -\dfrac{12}{13}$.

1.3 Solving by completing the square

A complete square is a quadratic that can be expressed in the form $(x \pm a)^2, a \in \mathbb{R}$.

To *complete a square* means to separate a quadratic expression into a complete square and a constant.

For example,

$$\underbrace{x^2 + 4x + 7}_{\text{quadratic}} = x^2 + 4x + 4 + 3 = \underbrace{(x+2)^2}_{\substack{\text{complete}\\ \text{square}}} + \underbrace{3}_{\text{constant}}$$

It is worth memorising the pattern of complete squares (these are obtained using equations (2) and (3)):

- $(x \pm 1)^2 = x^2 \pm 2x + 1$
- $(x \pm 2)^2 = x^2 \pm 4x + 4 \leftarrow$ this formula was used in the example above
- $(x \pm 3)^2 = x^2 \pm 6x + 9$
- $(x \pm 4)^2 = x^2 \pm 8x + 16$
- $(x \pm 5)^2 = x^2 \pm 10x + 25$
- ...

Example 13. Solve $x^2 - 6x + 8 = 0$ by completing the square.

Solution: Comparing the LHS expression with the 3rd equation from the pattern above, we can rewrite the equation as

$$x^2 - 6x + 9 - 1 = 0.$$

Note that it is the same equation, as $9 - 1 = 8$. Hence,

$$(x-3)^2 - 1 = 0;$$

$$(x-3)^2 = 1.$$

Using Example 2 we can see that $x - 3 = \pm 1$. The equation has two solutions: 4 and 2.

Answer: $x_1 = 4$, $x_2 = 2$.

Generally you can use the formula for completing the square:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c. \tag{10}$$

Please check this equality by applying the square of a sum formula (2) to its right-hand side.

Example 14. Solve $5x^2 + 3x - 2 = 0$ by completing the square.

Solution: We first divide both sides of the equation by 5 to have $a = 1$:

$$x^2 + \frac{3}{5}x - \frac{2}{5} = 0$$

We can now apply formula (10) to the expression in square brackets (noting that $b = 3/5$, $c = -2/5$).

$$x^2 + \frac{3}{5}x - \frac{2}{5} = \left(x + \frac{3}{10}\right)^2 - \frac{9}{100} - \frac{2}{5} = \left(x + \frac{3}{10}\right)^2 - \frac{49}{100}.$$

The equation becomes:

$$\left(x + \frac{3}{10}\right)^2 = \frac{49}{100}.$$

Hence

$$x + \frac{3}{10} = \pm\frac{7}{10}.$$

Answer: $x_1 = \frac{2}{5}$, $x_2 = -1$.

Problem 10 (6 marks). Solve the following quadratic equations by completing the square.

a) $x^2 + 8x + 7 = 0$; b) $x^2 - 14x + 13 = 0$;

c) $4x^2 - 8x - 21 = 0$; d) $11x^2 + 13x - 24 = 0$.

e) $x^2 - 4x = 0$; f) $x^2 - 10x + 18 = 0$.

Problem 11 (2 marks).

Show that formula (10) can be generalized to the case $a \neq 1$ as follows:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \tag{11}$$

Problem 12* (4 marks). Solve the following equation: $(x-3)^4 - 2x^2 + 12x - 81 = 0$

1.4 Solving using the discriminant formula

The solutions to a general quadratic equation are given by the well-known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{12}$$

You can obtain it from Eq. (11). A derivation can also be found in this Khan Academy video.

Read about this method on Isaac Physics and review the example therein.

In Eq. (12), the quantity $D = b^2 - 4ac$ is known as the *discriminant*. It determines the number of roots of the equation:

- if $D > 0$, the quadratic equation has two distinct real roots $x_1 = \dfrac{-b + \sqrt{D}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{D}}{2a}$.
- if $D = 0$, the quadratic equation has one distinct real root $x_1 = \dfrac{-b}{2a}$.
- if $D < 0$, the quadratic equation has no real roots.

Please note that in school, when using the formula, you are normally asked to find the answer to 3 s.f.; in this assignment please always give the exact answer, as shown in the following example:

Example 15. Find the exact roots of the equation $x^2 + 6x - 3 = 0$.

Solution: Using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times (-3)}}{2 \times 1} = \frac{-6 \pm \sqrt{36 + 12}}{2} = \frac{-6 \pm \sqrt{48}}{2}$$

This can be simplified:

$$x = \frac{-6 \pm 4\sqrt{3}}{2} = -3 \pm 2\sqrt{3}$$

Note that the answer is given in exact (surd) form.

Answer: $x_1 = -3 + 2\sqrt{3}$, $x_2 = -3 - 2\sqrt{3}$.

Problem 13 (5 marks). For the following quadratic equations:

1) find the discriminant and determine the number of real roots;

2) find the exact value of the roots.

a)$x^2 + 8x + 1 = 0$; b) $x^2 - 3x - 13 = 0$;

c) $5x^2 - 3x + 7 = 0$; d) $16x^2 + 8x + 1 = 0$.

e) $11x^2 - 5x = 0$;

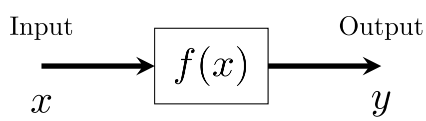
Problem 14* (4 marks). Find all the values of q such that equation

a) $qx^2 - x + q = 0$; b)$qx^2 - (q+1)x + 2q - 1 = 0$

has exactly one root.

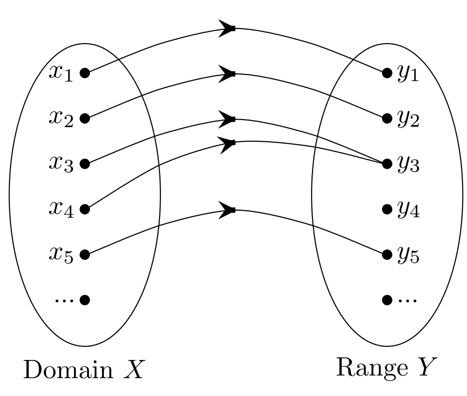
2 The Quadratic Function

A *numerical function* or simply a *function* is a mapping where one set of numbers is mapped onto another.



The function transforms x into y (we say $y = f(x)$). So the function is also a *rule* of transforming the input x into the output y.

The set of all possible values of the input is called the *Domain* and the set of all possible outputs is called the *Range*.



You can find more information on functions on this Khan Academy webpage.

The function in the form $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$ is known as the quadratic function. The graph of $y = f(x)$ is a *parabola.*

Please watch the videos on parabolas on this Khan Academy page. Some further information about quadratic functions and parabolas can be found on the AMSI website.

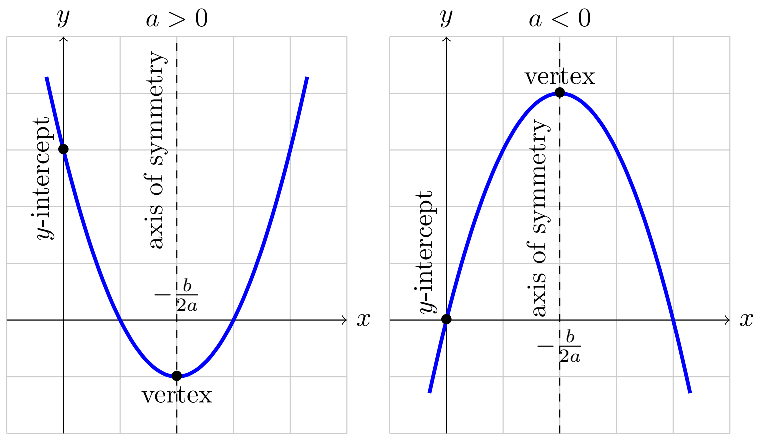


Figure 1: Salient features of a parabola.

The main features of the parabola are as follows.

- The *y-intercept* is the point where the parabola crosses the y-axis, i.e. $(0, f(0)) = (0, c)$,
- The *x-intercepts* are the points where the parabola crosses the x-axis. Their x-coordinates are the two roots (12) of the equation $ax^2 + bx + c = 0$.
- The *vertex* is the extremal (highest or lowest) point of the parabola. The x-coordinate of the vertex can be found by looking at Eq. (11). The squared term in parenthesis is never less then zero, hence the point $x = -\dfrac{b}{2a}$ where it *is* zero is the x-coordinate of the vertex.
- The parabola is *symmetric* about the vertical axis through the vertex. In particular, the x-coordinate of the vertex is located in the middle between the two x-intercepts: $-\dfrac{b}{2a} = \dfrac{x_1 + x_2}{2}$. You can see this from the formula (12).

Example 16. Find a,b and c if $M(-1, -7)$ is the vertex and $N(0, -4)$ is the y-intercept of the parabola $y = ax^2 + bx + c$.

Solution: The $x-$coordinate of the vertex is -1, so $-\dfrac{b}{2a} = -1 \Rightarrow b = 2a$. $(0, -4)$ is the $y-$intercept, so when $x = 0$, y is $-4 \Rightarrow -4 = a \times 0^2 + b \times 0 + c \Rightarrow c = -4$.

We also know that when $x = -1$, y is -7, so $-7 = a \times (-1)^2 + 2a \times (-1) - 4 \Rightarrow a = 3$. Hence, $b = 2a = 6$.

Answer: $y = 3x^2 + 6x - 4$, $a = 3$, $b = 6$, $c = -4$.

Problem 15 (2 marks). Find the parameters a, b, c of the two parabolas in Fig. 1. Assume that the grid step in the figure is 1.

Problem 16 (3 marks). Find the parameters a, b, c of the function $f(x) = ax^2 + bx + c$ if the graph $y = f(x)$ passes through the point $B(-1, -6)$ and has a vertex at $A(3, 2)$.

Problem 17 (modified PAT 2007, 3 marks). Sketch the curves $y = x^2$, $y = (x-3)^2$ and $y = (x-3)^2 - 7$ on the same set of axes.

Regular or decimal fractions?

Suppose you are given the equation $24x = 6$. Should your answer be $x = 1/4$ or $x = 0.25$?

While the two numbers are formally identical, a decimal fraction traditionally implies limited precision — that you don't know the answer beyond the significant figures given. In maths problems, this is usually not the case: the numbers are exact. To indicate this, you should use a regular fraction, writing $x = 1/4$.

In physics, in contrast, numbers are not known precisely. Consider the question: *A tortoise walked 6 meters in 24 seconds. What has been its speed?* Even though this is seemingly the same problem, you should answer "0.25 m/s", to implicitly indicate the precision, with which the answer is known. In theoretical equations and symbolic answers, however, you should still use regular fractions. For example, you should write $s = \frac{1}{2}at^2$ rather than $s = 0.5at^2$ — because $\frac{1}{2}$ here is an exact value.

Generally you should avoid mixing regular and decimal fractions. Answers should not contain decimals under a square root.

Functions are widely used in physics.

Problem 18 (4 marks). An object is thrown upwards. The height above ground (in meters) can be modelled by the function $h(t) = 20 + 12t - 5t^2$, where the time t is in seconds. Find:

a) the initial height above ground;

b) the maximum height reached by the object;

c) the times at which the object is 24 m above ground;

d) by finding the distance travelled in the first $0.1, 0.01, 0.001$ seconds, work out the initial velocity.