Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 10

Mathematics Assignment 02

Right-Angled Triangles & Trigonometry

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Due 17th November, 2024

This is the second Mathematics assignment from COMPOS. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in webinars on Tuesdays at 6pm, so think of the questions you would like to ask. Please submit what you have by the deadline, you are allowed to submit extra work after the review in tutorials. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 53 marks.

1 The Right-Angle Triangle

To begin, let's review the fundamental elements and properties of right-angled triangles.

A *right-angled triangle* is a type of triangle that has one angle measuring exactly 90 degrees. It is characterised by unique properties and relationships among its elements. The side opposite the right angle, which is also the longest side in a right-angled triangle, is called the *hypotenuse*. The two sides adjacent to the right angle are called *legs* or *catheti*.

One of the most famous theorems existing around right-angled triangle is the *Pythagoras' theorem*. It states that in any right-angled triangle, the square of the hypotenuse is equal to the sum of squares of the two legs. Mathematically, it can be expressed as: $c^2 = a^2 + b^2$ where c represents the length of the hypotenuse, and a and b represent the lengths of the legs. There are hundreds of ways to prove Pythagoras' theorem, some of which can be found on this brilliant.org page.

Problem 1 (2 marks). Show that in a right-angled triangle with a 30° angle, the shortest side is equal to half of the hypotenuse.

2 Basic Trigonometric Functions

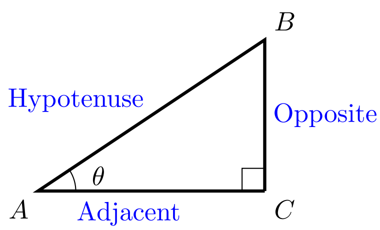
Trigonometry studies the relationships between angles and sides of right-angled triangles. The three primary trigonometric ratios are:

sine (sin): the ratio of the opposite side to the hypotenuse;

cosine (cos): the ratio of the adjacent side to the hypotenuse;

tangent (tan): the ratio of the opposite side to the adjacent side.

For a brief overview of these ratios you can watch this Khan Academy video.

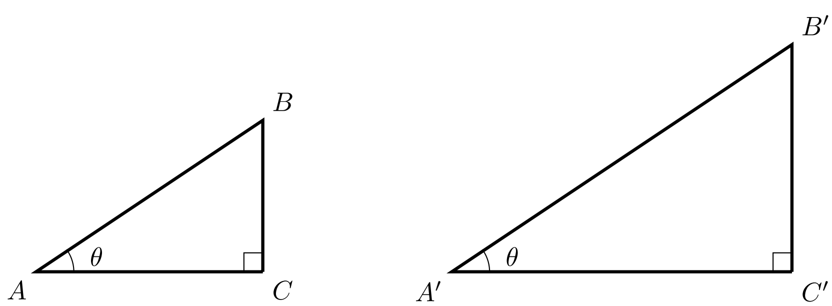


$$\sin\theta = \frac{BC}{AB}$$

$$\cos\theta = \frac{AC}{AB}$$

$$\tan\theta = \frac{BC}{AC}$$

A fundamental property of trigonometric functions is that they are dependent only on the value of the angle. This is because any two right-angled triangles with the same measure of the acute angles are similar to each other. For example, if in triangles ABC and $A'B'C'$ the angles C and C' are $90°$ and the angles A and A' are equal to θ, these triangles are similar, so the ratios of the sides are the same, e.g. $\dfrac{BC}{AB} = \dfrac{B'C'}{A'B'} = \sin\theta$. In other words, if we enlarge the triangle, sin, cos and tan of the angles don't change.



Let us derive two useful formulas. We'll start with an important relationship between a sine, a cosine and a tangent of an angle.

$$\tan\theta = \frac{BC}{AC} = \frac{BC}{AB}\bigg/\frac{AC}{AB} = \frac{\sin\theta}{\cos\theta},$$

so

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}.}$$

Another formula can be easily obtained from the Pythagoras' theorem $BC^2+AC^2 = AB^2$ by simply dividing both parts by AB^2. We find

$$\frac{BC^2}{AB^2} + \frac{AC^2}{AB^2} = 1 \quad \Rightarrow$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1.}$$

This is a pivotal finding known as the *Pythagorean trigonometric identity*, and we will explore its applications further in this assignment.

Given that we can construct a right-angled triangle for any acute angle, we can assert that the above equations hold true for all angles between $0°$ and $90°$.

Problem 2 (2 marks). Find sines, cosines and tangents of every acute angle in a:

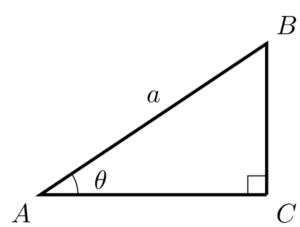
a) $30° - 60° - 90°$ right-angled triangle;

b) $45° - 45° - 90°$ right-angled triangle.

Do not use a calculator.

Example 1. In a right-angled triangle ABC ($\angle C = 90°$), $AB = a$ and $\angle A = \theta$. Find the lengths of the remaining two sides of the triangle in terms of a and θ.

Solution:



We know that $\sin\theta = \dfrac{BC}{AB} \Rightarrow BC = AB \times \sin\theta = a\sin\theta$

Similarly, $\cos\theta = \dfrac{AC}{AB} \Rightarrow AC = AB \times \cos\theta = a\cos\theta$.

Answer: $BC = a\sin\theta$; $AC = a\cos\theta$.

Problem 3 (2 marks). In a right-angled triangle ABC ($\angle C = 90°$), $BC = a$ and $\angle A = \theta$. Find the lengths of the remaining two sides of the triangle in terms of a and θ.

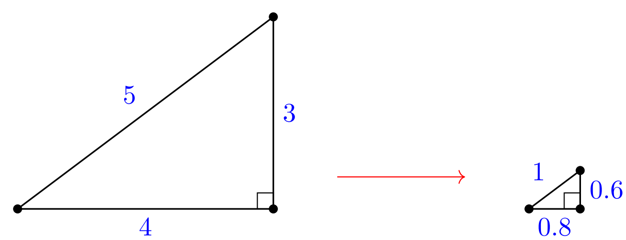
Problem 4 (2 marks). The apex angle of an isosceles triangle is β and its base length is a. Find the side lengths in terms of a and β.

3 Trigonometric Functions of Arbitrary Angles

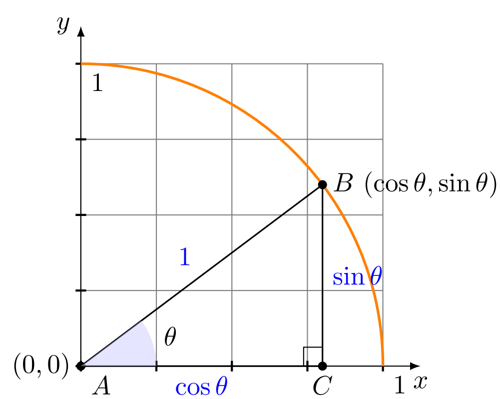
3.1 Trigonometry without triangles

In this chapter we will introduce a handy and informative instrument, helping us not only to memorise all the basic concepts of sines, cosines and tangents, but also to develop it further, beyond the geometry of right-angled triangles.

The first step is to recall that the sine, cosine and tangent are only dependent on the angle's magnitude and are hence equal in similar triangles. Therefore we can restrict our analysis to triangles with hypotenuse equal to 1. If we want to know the properties of trigonometric functions for any other triangle, we can simply find similar triangle with hypotenuse equal to 1. In other words, we can always scale any right-angled triangle so that the hypotenuse is 1.



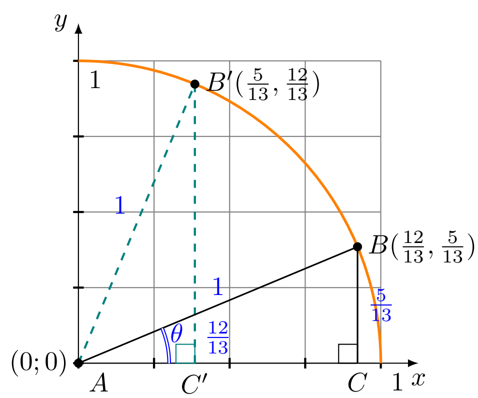
Consider a system of coordinates with the origin O. Let us place an arbitrary right-angled triangle ABC ($\angle C = 90°$) with hypotenuse $AB = 1$ so that A coincides with the origin and AC lies along the $x-$axis.



Let us now look at the angle $\angle CAB$, which we denote as θ. Its sine is equal to $\sin\theta = \dfrac{BC}{AB}$. But because $AB = 1$, the value of $\sin\theta$ is equal to the length of BC. Similarly we get that $\cos\theta$ is equal to the length of the leg AB. But we can also notice that, by construction, AC and BC are the $x-$ and $y-$ coordinates of vertex B. So, these coordinates are $(\cos\theta, \sin\theta)$.

Example 2. For a right-angled triangle with side lengths 5, 12, 13 draw its representative with hypotenuse of length 1 in the coordinate plane and hence find the sines and the cosines of the respective acute angles.

Solution. We have $5^2 + 12^2 = 25 + 144 = 169 = 13^2$, so this is indeed a right-angled triangle. The initial hypotenuse measures 13 units, so we should consider a triangle that is scaled down by a factor of 13. Therefore, the resulting side lengths are $\frac{5}{13}$ and $\frac{12}{13}$, which correspond to the sine and cosine of the smaller acute angle ($\triangle ABC$ in the figure below). If we are looking at the larger acute angle, we should swap the sides on the axes. Consequently, the sine and cosine values also switch places ($\triangle AB'C'$).



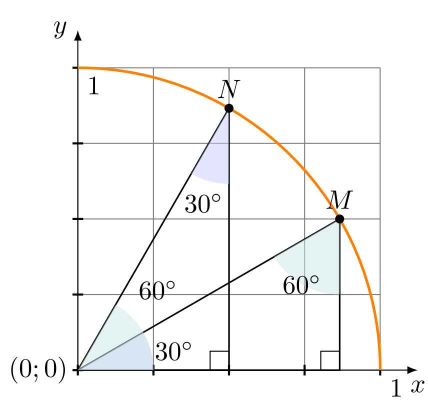
Answer:

$$\sin\angle BAC = \cos\angle ABC = \frac{5}{13};$$

$$\sin\angle ABC = \cos\angle BAC = \frac{12}{13}.$$

Note that in the above example we did not need to know the angle measures to find sin and cos.

Problem 5 (2 marks). Find the coordinates of the points M and N below.

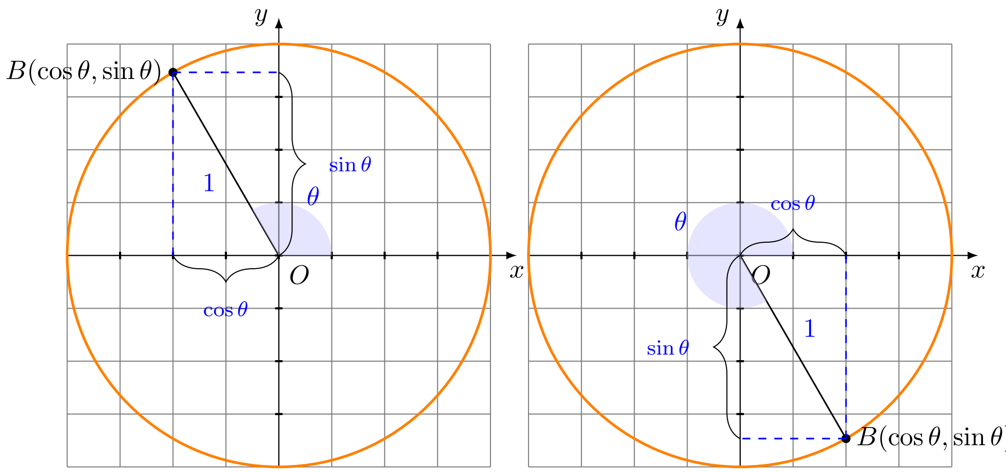


3.2 Trigonometric circle

Now, let's expand our understanding of sine, cosine, and tangent to angles beyond the range of $(0°, 90°)$. To do this, we'll consider a unit circle centred at origin, as shown in orange below. This circle is referred to as the *trigonometric circle*.

To find the trigonometric functions of a given angle θ, we draw a ray starting from the origin such that the angle between that ray and the positive $x-$axis is θ when measured *counterclockwise* from the $x-$axis. If B is the point of intersection between that ray and the trigonometric circle, then the coordinates of B are, similarly to the case of small angles considered above,

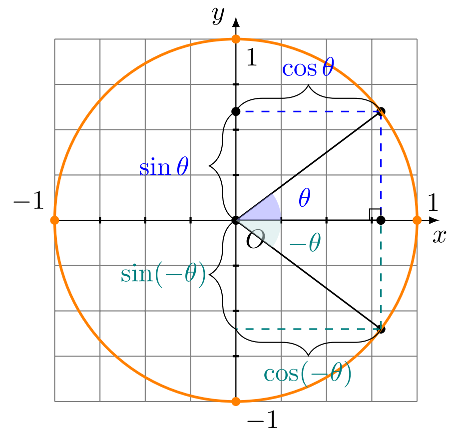
$$x = \cos\theta;$$
$$y = \sin\theta.$$



As a result, we can calculate sines and cosines for angles greater than $90°$, surpassing $180°$, and even exceeding $360°$. For the latter, it is worth noting that angles like $35°$ and $395°$ correspond to the same point on the circle. This is because they differ by $360°$, which is equivalent to one full rotation around the circle.

Moreover, one can define trigonometric functions for negative angles. To work with negative angles, we simply measure them in the same way as positive angles, but in a *clockwise* direction from the positive $x-$axis. As is evident from the figure below, we have

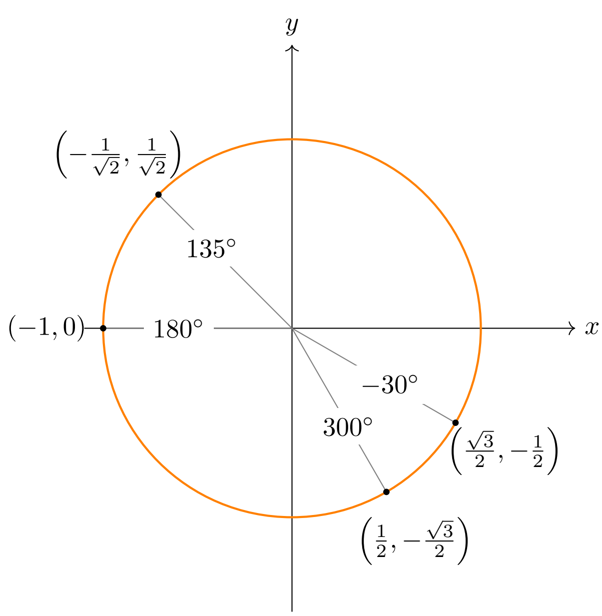
$$\sin(-\theta) = -\sin\theta;$$
$$\cos(-\theta) = \cos\theta.$$



Problem 6 (2 marks). For some angle θ, $\sin\theta = 4/5$ and $90^\circ \leq \theta \leq 180^\circ$. Find $\cos\theta$ and $\tan\theta$.

Example 3. Using the trigonometric circle, find the sine and cosine of the angles 135°, 180°, 300° and -30°.

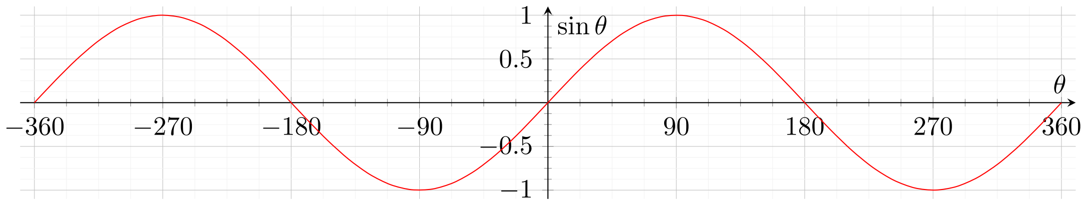
Solution. See below. For each angle, the coordinates of the point are $(\cos\theta, \sin\theta)$. For example, $\cos(-30^\circ) = \frac{\sqrt{3}}{2}$

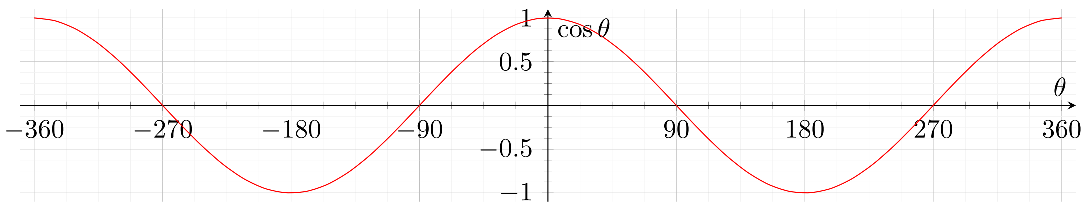


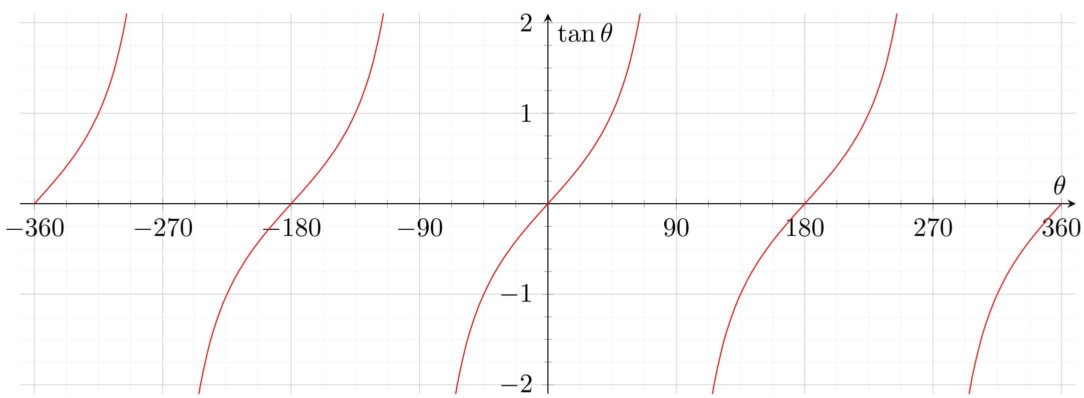
Problem 7 (2 marks). Complete the table below (all angles are in degrees).

	−180	−150	−135	−120	−90	−60	−45	−30	0	30	45	60	90	120	135	150	180
sin																	
cos																	
tan																	

Using the data from Problem 7, you should be able to plot the graphs of the three trigonometric functions. It is a good idea to do this by hand at least once.







From these plots we can see some properties of these functions:

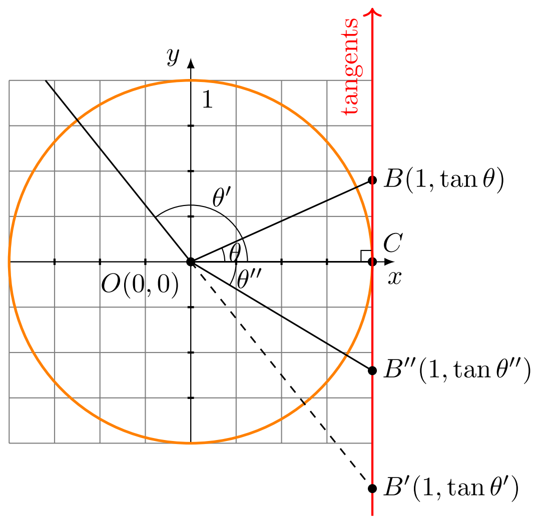
- sine and cosine are *periodic* functions with period $360°$;
- tangent is a periodic function with period $180°$.
- tangent is not defined for $\theta = -270°, -90°, 90°, 270°$ (this is because $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cos\theta = 0$ at these points);
- cosine and sine curves can be converted into each other by a horizontal shift through $90°$.
- sine and tangent are *odd* functions: $\sin(-\theta) = -\sin\theta$ and $\tan(-\theta) = -\tan\theta$;

- cosine is an *even* function: $\cos(-\theta) = \cos\theta$.

We will study these notions in more detail in our future assignments on functions.

3.3 The tangent axis

We found that the sine and cosine are interpreted as the coordinates of the corresponding point on the trigonometric circle. But what about tangent? It turns out that tangent has a nice geometric counterpart. To obtain it, we should consider a set of right-angle triangles whose *adjacent leg*, rather than hypotenuse, has length 1.



Examples are shown in the figure above. In triangle OBC, the leg OC adjacent to $\angle BOC = \theta$ has length 1, and hence $\tan\theta = \dfrac{BC}{OC} = BC$, which is equal to the $y-$coordinate of point B. We can hence call the vertical line at $x = 1$ — the extension of segment BC — the *tangent axis*: the $y-$coordinate of the intersection between that axis and the ray defining the angle equals the tangent of that angle.

As we can see, the tangent axis also works for angles outside the range $[0, 90°]$, including negative angles.

4 Trigonometric Transformations

In addition to sine, cosine and tangent, the following trigonometric functions are often used.

- cotangent $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$;
- secant $\sec\theta = \dfrac{1}{\cos\theta}$;
- cosecant $\mathrm{cosec}\ \theta = \dfrac{1}{\sin\theta}$.

Problem 8 (2 marks). By analogy with the tangent axis, introduce the *cotangent axis* such that a coordinate of a point on that axis is the cotangent of the corresponding angle.

Problem 9 (2 marks). Plot $\cot\theta$ for $-360° < \theta < 360°$. For which θ is this function not defined? What is the period of this function?

Example 4. Show that the following relation holds for all angles at which the relevant trigonometric functions are defined:

$$\sin^2\theta + \tan^2\theta = \sec^2\theta - \cos^2\theta.$$

Solution. Recalling that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$, we can rewrite the above relation as

$$\frac{\sin^2\theta\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1-\cos^4\theta}{\cos^2\theta}.$$

Multiplying both sides by $\cos^2\theta$, we rewrite this as

$$\sin^2\theta(1+\cos^2\theta) = 1-\cos^4\theta.$$

Next, we recall that $\cos^2\theta + \sin^2\theta = 1$, hence we can express $\sin^2\theta = 1-\cos^2\theta$. Now the relation becomes

$$(1-\cos^2\theta)(1+\cos^2\theta) = 1-\cos^4\theta.$$

This is a manifestation of the familiar identity $(a+b)(a-b) = a^2 - b^2$.

Problem 10* (7 marks). Show that the following relations hold for all angles at which the relevant trigonometric functions are defined:

a) $\dfrac{1+\cot\theta}{1-\cot\theta} = \dfrac{\tan\theta+1}{\tan\theta-1}$;

b) $\tan^2\theta - \sin^2\theta = \tan^2\theta\sin^2\theta$;

c)** $\sin^6\theta + \cos^6\theta - \dfrac{3}{4}\left(\dfrac{1}{\sec^2\theta} - \dfrac{1}{\mathrm{cosec}^2\theta}\right)^2 = \dfrac{1}{4}$.

Problem 11* (4 marks). Simplify the following expressions:

a) $\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta$;

b) $\dfrac{(1+\cot\theta)(\sec^2\theta-1)}{(1+\tan^2\theta)\mathrm{cosec}^2\theta}$.

Example 5. Let $\sin\theta + \cos\theta = 1.4$. Find $\sin\theta\cos\theta$.

Solution: We have $(\sin\theta+\cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1.4^2 = 1.96$. Now recalling that $\sin^2\theta + \cos^2\theta = 1$, we find $2\sin\theta\cos\theta = 0.96$, and hence $\sin\theta\cos\theta = 0.48$.

Problem 12 (2 marks). Let $\sin\theta + \cos\theta = A$. Express $\sin^3\theta + \cos^3\theta$ in terms of A.

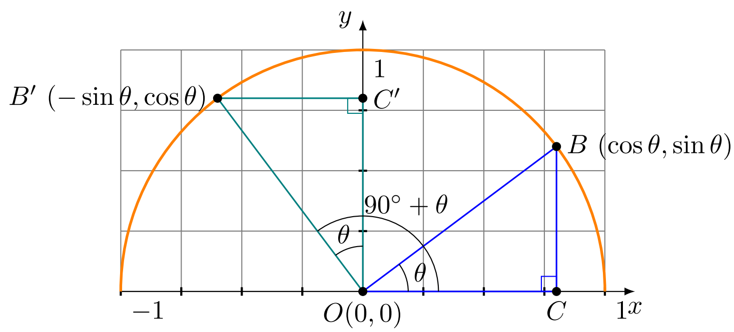
Problem 13 (2 marks). Express $\sin\theta$ and $\cos\theta$ $(0 \le \theta \le 90°)$ in terms of $\tan\theta$.

5 Trigonometric reduction formulae

In this section, we will study how to express trigonometric functions of large angles, such as $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$ through trigonometric functions of θ. This is useful in many calculations.

Example 6. Express $\sin(90° + \theta)$ and $\cos(90° + \theta)$, where $0 \le \theta \le 90°$, in terms of $\sin\theta$ and $\cos\theta$.

Solution. The answer is self-evident by inspecting the figure below, but let us make a rigorous argument.



We display both angles $\theta = \angle BOC$ and $90° + \theta = \angle B'OC$ on a coordinate system, with points B and B' lying on the trigonometric circle and point C being the projection of B onto the $x-$axis. Let us define the point C' as the projection of B' onto the $y-$axis and consider $\triangle B'OC'$. The angles $\angle BOC$ and $\angle B'OC'$ are both equal to θ, and hence the right-angled triangles $\triangle BOC$ and $\triangle B'OC'$ are congruent (because their hypotenuses are both of length 1). Therefore $BC = B'C'$ and $OC = OC'$. But we know that the length of $BC = \sin\theta$ is the $y-$coordinate of point B while the length of $B'C'$ is the negative $x-$coordinate of point B' and hence $B'C' = -\cos(90° + \theta)$. Similarly, $OC = \cos\theta$ is equal to OC', whose length is the y-coordinate of B' and hence $OC' = \sin(90° + \theta)$. We conclude that

$$\sin(90° + \theta) = \cos\theta;$$
$$\cos(90° + \theta) = -\sin\theta.$$

This result is also manifest from the plots of sine and cosine functions. Indeed, if we shift the sine curve to the left by $90°$, we obtain the cosine curve. And if we shift the cosine curve to the left by $90°$, we obtain the negative sine curve.

Problem 14* (5 points). Express the sines, cosines, tangents and cotangents of the angles $(90° - \theta)$, $(180° \pm \theta)$ and $(-90° \pm \theta)$ in terms of trigonometric functions of angle $\theta \in [0, 90°]$. Make a drawing for each case.

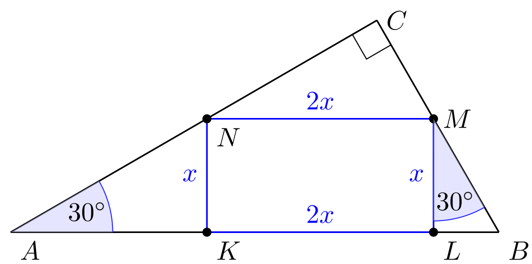
Note that, although we derived the trigonometric reduction formulae assuming θ to be within the first quadrant of the trigonometric circle, these formulae apply for *any* real θ.

6 Applications of Trigonometry in Geometry

We conclude this assignment with a few elegant geometric problems, where the use of trigonometry may (or may not) be helpful.

Example 7 (4 marks). In a right-angled triangle with a hypotenuse of length l and an acute angle of $30°$, a rectangle is inscribed. The length of the rectangle is double its width. The longer side of the rectangle lies on the hypotenuse, and two other vertices are positioned on the legs. Find the sides of the rectangle.

Solution.



Let us denote the short side of the rectangle as x, then the long side is $2x$. Defining the points as in the figure above, we find $AK = \dfrac{NK}{\tan 30°} = x\sqrt{3}$ and $LB = \dfrac{ML}{\tan 30°} = \dfrac{x}{\sqrt{3}}$. Hence $AB = AK + KL + LB = x\sqrt{3} + 2x + \dfrac{x}{\sqrt{3}} = x\dfrac{4+2\sqrt{3}}{\sqrt{3}}$. Because we are given that the hypotenuse $AB = l$, we find $x = \dfrac{\sqrt{3}}{4+2\sqrt{3}}l$. We can simplify this as

$$x = \frac{1}{2}\frac{\sqrt{3}}{2+\sqrt{3}}l = \frac{1}{2}\frac{\sqrt{3}}{2+\sqrt{3}}\frac{2-\sqrt{3}}{2-\sqrt{3}}l = \frac{1}{2}\frac{2\sqrt{3}-3}{2^2-\sqrt{3}^2}l = \frac{2\sqrt{3}-3}{2}l.$$

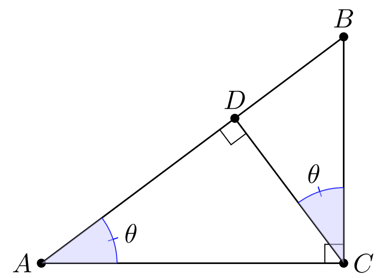
Problem 15 (3 marks). Two vertices of a square are positioned on the base of an isosceles triangle and two other vertices on its sides. Find the side of the square if the base of the isosceles triangle is a and its apex angle is θ.

Example 8. In a right-angled triangle ABC, an altitude CD drawn from the right angle to the hypotenuse divides the hypotenuse into two segments AD and BD. Show that the length of the altitude is the geometric mean of the lengths of these two segments, i.e. $CD = \sqrt{AD \cdot BD}$.

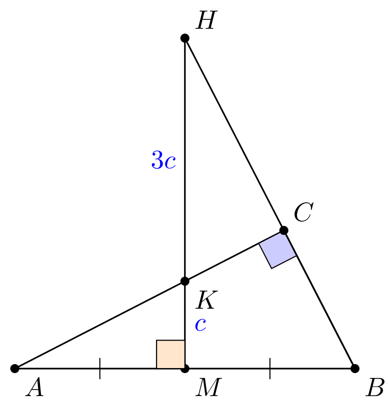
Solution. Let us express the tangents of equal angles $\angle DAC = \angle DCB = \theta$ in the right-angled triangles DAC and DCB correspondingly:

$$\tan\theta = \frac{DC}{AD} = \frac{BD}{DC}$$

Thus, $DC^2 = AD \times BD$. $\square$



Problem 16 (4 marks). In a right-angled triangle ABC, a perpendicular is drawn from the midpoint M of the hypotenuse AB. The segment KM of this perpendicular contained within the triangle is equal to c, and the segment KH, where H is the point of intersection of the extensions of BC and KM, is equal to $3c$. Find the length of the hypotenuse AB.



Problem 17 (4 marks). One side of a triangle has length 2, its adjacent angles are $30°$ and $45°$. Find the two other sides of the triangle.

Problem 18 (4 marks).

a) The hypotenuse of a right-angled triangle has length a. Find the length of the segment connecting the apex of the triangle's right angle and the middle of the hypotenuse[1].

b) A kitten is sitting in the middle of a ladder leaning against a wall. The ladder starts to slide along the wall and the floor. What is the trajectory of the kitten's movement?

[1] A segment connecting one of the triangle's angles and the middle of the opposite side is called a *median*.