Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 11

Mathematics Assignment 01

Combinatorics and Probabilities

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This is the first Mathematics assignment from COMPOS for Year 11. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections.

We hope that eventually you will be able to solve most of the problems. Good luck!

Total 54 marks.

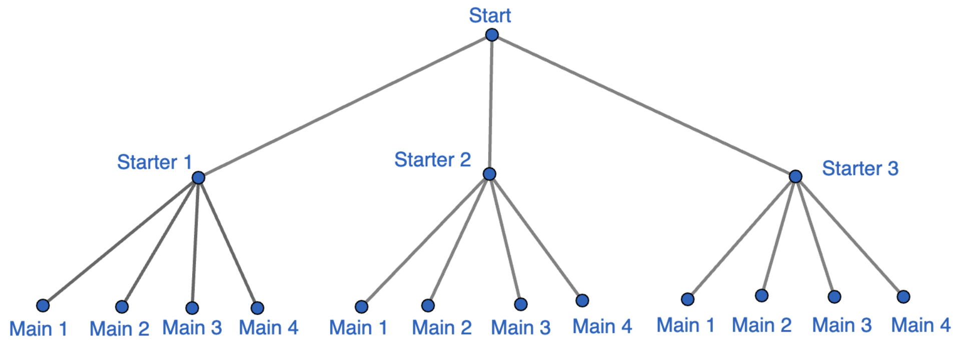
1 The Multiplication Principle

Counting problems, or *combinatorics*, is perhaps the branch of Mathematics which is the purest form of problem solving. There is little in the way of theoretical material to learn, and rather than memorising techniques it is often best to take each problem on its own merit. However, one idea that is worth mentioning is the *multiplication principle* which states that if there are m ways of doing something in the first step, and n ways of doing something in the second step, then there are mn choices overall.

Example 1. A restaurant menu consists of three starters and four mains. How many meals can customers choose?

Solution Each of the three starters can be paired up with any one of the four main courses. This gives 12 possibilities overall. This is an example of the multiplication principle in action.

This can be illustrated using the following diagram — the *tree diagram.* In the first step, we may choose any of the three edges, each of which will lead to another four choices.



It is helpful to use diagrams to illustrate your solutions at the beginning. However, it is important that you eventually start applying the method mentally.

This principle can be extended to more steps. If there are n_1 ways of doing something in the first step, n_2 ways of doing something in the second step, and so on until there are n_k ways of doing something in the kth step, then overall the number of ways overall is the product $n_1 n_2 n_3 \ldots n_k$.

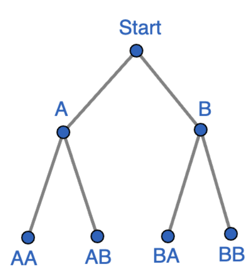
Example 2. The country of Binaria only has two letters, A and B.

a) How many two-letter words are there?

b) How many k-letter words are there?

Solution.

a) There are two choices for the first letter (either A or B) and each of these choices can be followed by one of two letters (again either A or B) to form $2 \times 2 = 4$ words in total.



b) Extending the idea of part a) we can conclude that there are 2^k words, as there are 2 choices at each stage.

Another approach to this problem is to show that the number of words doubles each time we increase the number of letters by one because we can form words of length $n+1$ by adding either an A or a B to the end of each of the words of length n. We could also map the n letter words in Binaria onto binary numbers from 0 to $2^n - 1$ in by replacing A with 0 and B with 1. There are clearly 2^n binary numbers in this range.

Example 3. How many 7-digit numbers contain at least one seven as a digit?

Solution. Firstly let us think how many seven-digit numbers there are.

The leading digit of a seven digit number cannot be zero, so there are only nine choices $(1, 2, 3, \dots, 9)$. All other digits could be zero, so there are 10 choices for the remaining six digits. Using the multiplication principle we have that there are 9×10^6 seven digits numbers. Notice we would get exactly the same result if we simply thought about the number of numbers between $1,000,000$ and $9,999,999$ (inclusive), which is just $10^7 - 10^6$.

We want to count those seven digit numbers with a seven. It turns out that it is easier to count the number of seven digit numbers *without* a seven. If none of the digits can be a seven then there are only eight choices for the leading order digits, and nine choices for the remaining six digits. Then using the multiplication principle we have that there are 8×9^6 digits without a seven.

The number of seven digit numbers which contain a seven is hence $9 \times 10^6 - 8 \times 9^6$. It is possible to evaluate this number as $4,748,472$, but in many ways the answer $9 \times 10^6 - 8 \times 9^6$ is preferable as it is more suggestive of the method.

Problem 1 (1 mark). A coin is tossed three times. How many different sequences of heads and tails can be obtained? Draw a tree diagram.

Problem 2 (3 marks). In Wonderland, there are three cities: A, B, and C. There are 6 roads from city A to city B, and 4 roads from city B to city C.

a) How many different ways are there to travel from A to C?"

b) A new city D has been built along with several new roads – two from A to D and two from D to C. How many different ways are there now to travel from city A to city C?

Problem 3 (PAT 2016, 3 marks). How many numbers greater than 5000 may be formed by using some or all of the digits 3, 4, 5, 6, and 7 (but only those) without repetition?

2 Permutations and arrangements

A *permutation* is a way of arranging a set of objects *in order*. For example, there are six ways of putting the letters ABC into order (ABC, ACB, BAC, BCA, CAB and CBA). The following example shows how we can count the number of permutations of a set by using the multiplication principle.

Example 4. A greyhound race consists of six dogs. Assuming there are no dead heats (events when two or more dogs cross the finish line at exactly the same time), in how many possible orders can the six dogs finish in?

Solution.

Any one of the six dogs could win. Once the winner has been selected, any one of the remaining five dogs could finish second, and then any one of the remaining four dogs could finish third, and so on until there is only one remaining choice for the dog finishing last.

Using the multiplication principle, we can conclude that there are

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

possible orders for the finishing order of the race.

The previous example can be generalised, to show that there are $n \times (n-1) \times (n-2) \ldots 1$ ways of putting n objects into order. This product is normally written as $n!$.

The following example is a generalisation of permutations where only some of the elements in the set are included.

Example 5. The Permian alphabet has the same twenty-six letters as the English alphabet. Permian words consist of any possible string of five letters *without* repeats. How many Permian words are there?

Solution The first letter of the word could be any one of the twenty-six letters of the alphabet. The second letter because the second letter needs to be different to the first letter, so only twenty-five choices remain. Similarly, after choosing the first two letters there are only 24 choices remaining for the third letter, and so on we are left with only 22 choices for the fifth and final letter. This means that there are $26 \times 25 \times 24 \times 23 \times 22 = 7,893,600$ Permian words.

This example can be generalised, to show that the number of ways, ${}^nP_k$, of choosing r selections in order from choice of n possibilities is given by the formula:

$${}^nP_r = \frac{n!}{(n-r)!}.$$

We will refer to this quantity as the number of *arrangements* of r items from n objects[1]. Obviously, the special case when $n = r$ is just the total number of permutations of n objects.

In this Khan Academy video, the results obtained above are explained once again. You can watch it if needed.

Problem 4 (1 mark). In a group of 11 friends, a team leader and an assistant leader need to be chosen. How many different ways can this be done?

Problem 5 (2 marks). How many odd six digit numbers can be made using the digits $1, 2, 4, 5, 6$ and 8?

a) If digits can be repeated.

b) If digits cannot be repeated.

Problem 6 (2 marks). There are fifteen books on a bookshelf. Five are red, five are green and five are blue. How many permutations of the books are possible if all books of the same colour must be kept together?

[1] ${}^nP_r$ is also sometimes called a *partial permutation* or a *k-permutation*.

3 Combinations

Sometimes the order in which objects are selected does not matter. For example, in most card games players are allowed to reorder the cards that are dealt to them. In this video by Khan Academy you will be introduced with the concept of *combinations*.

The number ${}^{n}C_r$ represents the number of ways of selecting r different objects from a choice of n possible objects, where the order of the objects does not matter. Let us obtain a formula for ${}^{n}C_r$ by thinking by generalising the following example.

Example 6. The Combian alphabet has the same twenty-six letters as the English language. Combian words consist of all five letter words with five different letters which are written in alphabetical order. So the string $ACERX$ is a word, but $AECRX$ is not a word as the letters are not in alphabetical order. How many words are there in the Combian language?

Solution.

If we consider the strings consisting of letters A, C, E, R and X, we know, from our discussion of permutations, that there are $5! = 120$ ways of putting the letters into order. Of these 120 permutations only one will be a Combian word, but all 120 will be a Permian word (see Example 4).

We saw in Example 4 that there were $\frac{26!}{21!}$ Permian words, so the number of Combian words is given by

$$\frac{26!}{21!5!} = 65,780$$

The number of Combian words is simply the number of ways of choosing five letters from an alphabet of twenty-six letters, so we could have written the answer as ${}^{26}C_5$. Generalizing our reasoning we can deduce that

$${}^{n}C_r = \frac{{}^{n}P_r}{r!} = \frac{n!}{(n-r)!r!}. \tag{1}$$

We refer to this quantity as the number of *combinations* of r items from n objects. To reiterate, the difference between the number of combinations and the number of arrangements, ${}^{n}P_r$, is that for combinations the order of the objects does not matter. For example, in the British National Lottery players win if they choose the same six numbers that are pulled out by the lottery machine. They do not need to specify the order that the balls are chosen, so the probability that they win is $\frac{1}{{}^{52}C_6}$ rather than $\frac{1}{{}^{52}P_6}$. Not needing to choose the balls in order makes winning the lottery easier, but because ${}^{52}C_6$ is anyway over fourteen million, you are still very unlikely to win.

Example 7. A three digit number is said to be *good* if its digits are non-decreasing, for example $333, 446, 288$ and 356 are all *good*. The number 217 is not good because 1 is strictly smaller than 2. How many *good* numbers are there?

Solution This is a difficult problem, as the fact that the digits can be repeated means that it does not obviously fall into the category of counting problem covered by combinations or permutations. There is however a neat trick.

We can take any good number and add one to the second digit, and two to the third digit (possibly getting numbers above 9) to get a three digit number with three strictly different digits. For example 458 becomes

$\{4, 6, 10\}$. Conversely if we take any three different numbers between 1 and 11 we can turn it into a *good* number by forming a string made up of the smallest number, one less than the second smallest number and two less than the largest number, for example $\{2, 3, 11\}$ becomes 229. In other words, there is *one-to-one correspondence* between the set of *good* numbers and the set of monotonically-increasing combinations of three numbers chosen from $\{1, 2, 3, \ldots, 11\}$. This means that the two sets have the same number of elements. The number of elements in the latter set is easy to compute: it is so the answer is ${}^{11}C_3 = 165$.

Problem 7 (3 marks). The coach has to pick a team of one goalkeeper, four defenders, four midfielders and two attackers from a squad of 22 player. Two teams are the same if they have the same goalkeeper, the same four players in defence, the same four players in midfield and the same two attackers. The order of the players within the defence, for example, does not matter. In how many ways is this possible?

Example 8. There are six ways of writing 5 as a sum of three non-zero natural numbers. Here, the same numbers in different orders count as different sums.

$$
\begin{aligned}
5 &= 1 + 1 + 3 \\
5 &= 1 + 2 + 1 \\
5 &= 1 + 3 + 1 \\
5 &= 2 + 1 + 2 \\
5 &= 2 + 2 + 1 \\
5 &= 3 + 1 + 1
\end{aligned}
$$

How many ways are there of writing n as a sum of three non-zero natural numbers?

Solution. Let us represent the number n as a row of n circles. Two dividers placed into the row split it into three parts, the sum of which is n. For example, $(\bullet|\bullet\bullet\bullet|\bullet)$ represents $5 = 1 + 3 + 1$. There is a total of $n - 1$ places where the dividers can be positioned, of which two must be chosen. Hence the answer is ${}^{n-1}C_2$. For example, if $n = 5$, ${}^{n-1}C_2 = 6$.

Problem 8 [*. (5 marks)] An ice cream shop sells three flavours of ice cream (vanilla, chocolate and strawberry) and customers are allowed to order as many scoops of the same flavour as they want. In how many ways is it possible to order an ice cream consisting of three scoops

a) if the order of the scoops matters;

b) if the order of the scoops does not matter?

Can you generalise to n scoops of m possible flavours?

4 Pascal's Triangle

Example 9. Prove the following:

$${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Solution The result can be proved algebraically by simply using the formula (1) to get expressions for and simplifying the fraction:

$$^{n-1}C_{r-1} + ^{n-1}C_r = \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}.$$

This is left as an exercise for the reader. But let us consider an example to get an intuition for this result. Suppose a football coach has a squad of n players and he needs to pick a team of r players. The star player has picked up a slight injury and the coach is unsure if he can play.

If the star player is fit, then the coach will play him. This will leave him $r-1$ players to select, from the remaining $n-1$ players in the squad. He can do this in $^{n-1}C_{r-1}$ ways.

If the star player is not fit, then the football coach will need to pick the entire team of r players from the remaining $n-1$ players in the squad. He can do this in $^{n-1}C_r$ ways.

Adding these two together the number of ways in each of these two scenarios gives that there are $^{n-1}C_{r-1} + ^{n-1}C_r$ ways of picking the team. However, we could also have simply said that the number of ways of selecting r players from a squad of n is simply ^nC_r. This proves the result.

The triangle of numbers whose nth row consists of the numbers $^nC_0, ^nC_1, ^nC_2 \ldots ^nC_n$ is known as Pascal's Triangle after the 17th century French scientist Blaise Pascal. However, it was first discovered much earlier by the Indian Mathematician Pingala (ca. 200 BCE) and in China it is often referred to as Yang Hui's Triangle after the Chinese Mathematician who studied the triangle in the 13th century.

The first few rows of Pascal's Triangle are as follows.

$n=0$							1						
$n=1$						1		1					
$n=2$					1		2		1				
$n=3$				1		3		3		1			
$n=4$			1		4		6		4		1		
$n=5$		1		5		10		10		5		1	
$n=6$	1		6		15		20		15		6		1

As is evident from Example 9, the numbers in Pascal's Triangle are formed by the sum of the two adjacent numbers in the row above.

Pascal's Triangle has many more astonishing properties and hidden mathematical patterns, connected to the properties of combinations that we derived in the previous chapter. You can learn many of them in this comprehensive video by *What About Why*.

Problem 9 (3 marks).

a) Calculate the sum of all numbers in each of the first seven rows of Pascal's Triangle. What are these sums?

b) Generalize the formula for the sum of all numbers in a row and prove it. *Hint:* Start indexing rows from 0, i.e., the 1st row is the one with 1 1.

5 Binomial Theorem

As some of you may know, the numbers in Pascal's triangle also occur in the Binomial Theorem. This can be understood in the spirit of combinatorics. Let us write, for example,

$$\begin{aligned}(x+y)^4 &= xxxx + xxxy + xxyx + xxyy + xyxx + xyxy + xyyx + xyyy \\ &\quad + yxxx + yxxy + yxyx + yxyy + yyxx + yyxy + yyyx + yyyy.\end{aligned}$$

The number of terms similar to e.g. $xxyy$ equals to the number of ways we can pick two positions in the four-letter word that would be occupied by the letter x. There are ${}^4C_2 = 6$ such ways:

$$xxyy, xyxy, xyyx, yxxy, yxyx, yyxx.$$

More generally, the expression $(x+y)^n$ will consist of all the words of n letters made up of xs and ys. Of these the number of words with r lots of x and $n-r$ lots of y (in some order) is ${}^nC_r$. We hence have the familiar binomial expansion:

$$(x+y)^n = x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}y^2 + \cdots + y^n.$$

Problem 10 (PAT 2014, 3 marks). What is the coefficient of x^7 in the expansion of $(1+2x)^4(1-2x)^6$?

6 Probability

The counting ideas we have met above are often used in probability problems, especially *discrete problems* and which have a finite number of equally likely outcomes. In these cases we just need to count the number of outcomes that satisfy the condition we are interested in, and divide by the total number of outcomes.

Example 10. Ten people, including Alice and Bob, go to the cinema. They sit in a row of ten seats in a random order. What is the probability that Alice and Bob sit next to one another?

Solution.

Let us count the permutations in which Alice and Bob sit next to one another. We can do this by firstly imagining that Alice and Bob are put together in a sack. We now consider the sack (containing Alice and Bob) and the eight other people, then we have 9 objects in total, which can be ordered in 9! ways. When we open the sack to release Alice and Bob we can order them in 2! ways, using the multiplication principle, this gives $2 \times 9!$ orders in which Alice and Bob are sitting next to one another.

The number of ways that the ten friends can order themselves in the cinema is simply 10!, so the desired probability is

$$\frac{2 \times 9!}{10!} = \frac{1}{5}$$

The counting ideas we have seen in this sheet can make systematic listing of outcomes more efficient.

Example 11. Three standard dice are rolled, what is the probability the total is 15?

Solution

Let us first count the dice roll outcomes that would lead to a total of 15. The sets of three numbers between 1 and 6 which add up 15 are $\{6, 6, 3\}$, $\{6, 5, 4\}$ and $\{5, 5, 5\}$.

Of these, $\{6, 6, 3\}$ can occur in ${}^3C_1 = 3$ ways (choose the one place for the 3). $\{6, 5, 4\}$ can occur in six ways (3! orders of the three different numbers) and $\{5, 5, 5\}$ can occur only once. This gives a total of 10 ways.

Since each die can have one of six possible outcomes there are a total of $6 \times 6 \times 6 = 216$ possibilities. So the required probability is $\frac{10}{216} = \frac{5}{123}$.

Problem 11 (3 marks). In the city where the Absent-Minded Scientist lives, phone numbers consist of 7 digits. The Scientist easily remembers a phone number if it is a palindrome, meaning it reads the same from left to right and right to left. For example, the number 4435344 is easy for the Scientist to remember because it is a palindrome. However, the number 3723627 is not a palindrome, so the Scientist has difficulty remembering it. Find the probability that the phone number of a new random acquaintance will be easy for the Scientist to remember.

Problem 12 (5 marks). 23 people are selected at random. Find the probability that at least 2 of them share a birthday (day and month).

Problem 13 (PAT 2006, 3 marks). Three dice are thrown, one after the other. Calculate the probability that

(i) all three dice give a six;

(ii) all three dice give the same number;

(iii) only the third die gives a six.

Problem 14 (PAT 2008, 4 marks).

A die is biased so that the numbers 5 and 6 are obtained three times as often as 2, 3 and 4, and the number 1 is never obtained. Calculate the probability that (i) a two is thrown; (ii) two consecutive throws give a total ≥ 10.

Problem 15 (PAT 2009, 4 marks).

If two identical dice are thrown, what is the probability that the total of the numbers is 10 or higher? [Hint: list the combinations that can give a total of 10 or higher.]

Two dice have been thrown, giving a total of at least 10. What is the probability that the throw of a third die will bring the total of the three numbers shown to 15 or higher?

Problem 16 (PAT 2010, 3 marks).

In a game of dice, a player initially throws a single die, and receives the number of points shown. If the die shows a 6, the player then throws the die again and adds the number shown to his/her score. The player does not throw the die more than twice. Calculate the probability that the player will gain an even number of points.

Problem 17 (PAT 2012, 3 marks).

Consider two dice – one contains the numbers 1–6, the other contains only 1,2,3 each shown twice (i.e. 1,2,3,1,2,3). What is the probability that when we roll the two dice we will obtain a score of 7?

Problem 18 (PAT 2015, 3 marks).

An unbiased coin is tossed 3 times. Each toss results in a "head" or "tail". What is the probability

(a) of two or more tails in succession?

(b) that two consecutive toss results are the same?

(c) that if any one of the toss results is known to be a tail, that all of the tosses resulted in tails?