Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 11

Mathematics Assignment 02

Basic Differentiation

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This is the second Mathematics assignment from COMPOS Y11. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in tutorials and webinars, so think of the questions you would like to ask. Please submit what you have by the deadline. We hope that eventually you will be able to solve most of the problems. Good luck!

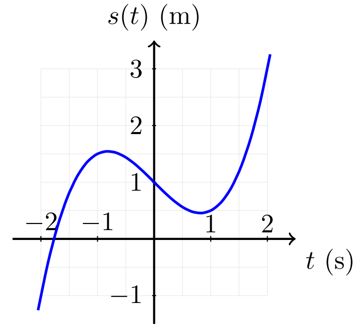
Total 35 marks.

The aim of this assignment is to learn about *differentiation*, or finding the *derivative* of a function. Differential calculus is a powerful tool for solving problems, especially in physics. We will practice differentiation using power and trigonometric functions only, leaving exponential and logarithmic functions for future assignments.

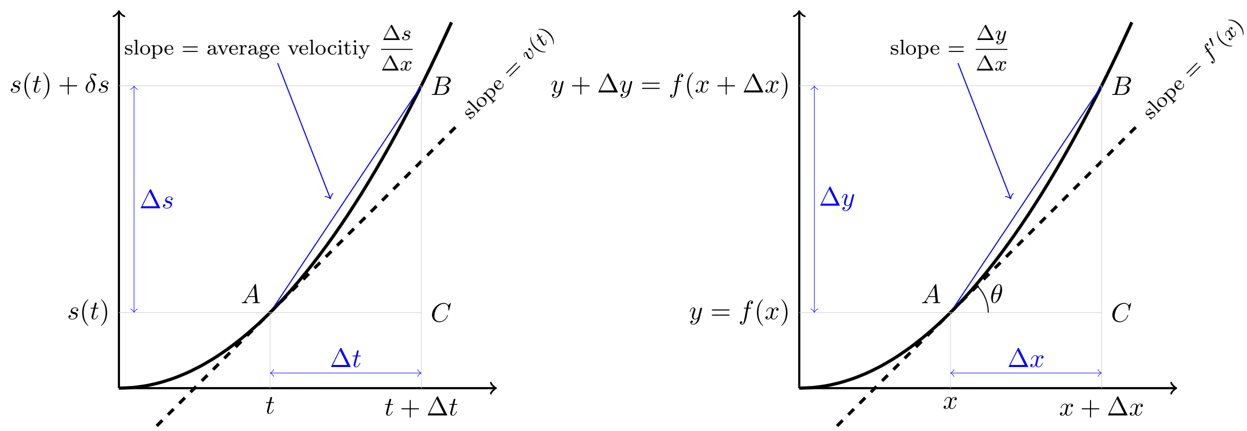
1 What is differentiation?

Although we have not yet introduced the derivative formally, we have already encountered it implicitly in Y10 Physics Assignment 3 on 1D motion, in which we learned the notion of *instantaneous velocity*. Instantaneous velocity $v(t)$ is the derivative of the coordinate $x(t)$ or displacement $s(t)$. Before proceeding, please take a moment to review Section 2.3 of that assignment and the example therein.

Problem 1 (3 marks). For the plot of $s(t)$ given below, estimate the instantaneous velocity $v(t)$ at several points in time and hence sketch its graph (use the template in the Appendix).



The derivative is the generalization of the instantaneous velocity to an arbitrary function $y = f(x)$: it is the *instantaneous rate of change* of that function. The analogy is shown in the figure below. For an arbitrary point $A = (x, y)$ on the graph of $f(x)$, the rate of change is approximated by $\dfrac{\Delta y}{\Delta x}$, where $B = (x+\Delta x, y+\Delta y)$ is another point on that graph. The smaller Δx, the closer B to A, the more accurately does this ratio approximate the instantaneous rate of change of $f(x)$. Of course, we cannot set Δx to exact 0, because the fraction $\dfrac{0}{0}$ is not defined. However, for any other Δx, however small, this fraction does exist.



Mathematically, we write this as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \tag{1}$$

where the symbol $\lim_{\Delta x \to 0}$ means the value that $\dfrac{\Delta y}{\Delta x}$ approaches when Δx approaches zero.

We will also use alternative notations for the derivative, e.g. $f'(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$. The symbol "d" here can be interpreted as an infinitely small increment of x or y.

The slope of segment AB equals $\dfrac{\Delta y}{\Delta x} = \dfrac{BC}{AC} = \tan \angle BAC$. As point B approaches A, the line AB approaches the tangent to the graph of $f(x)$, so the derivative is equal to the slope of that tangent: $f'(x) = \tan\theta$.

At a given x, the derivative $f'(x)$ is equal to the slope of the tangent to the curve $y = f(x)$.

2 Power rule

Finding $f'(x)$ by explicitly calculating $f(x+\Delta x)$ and $f(x)$ is called *differentiation from the first principles*.

Example 1. Find the derivative $f'(x)$ from first principles of the functions

a) $f(x) = 1$;

b) $f(x) = 2x$;

c) $f(x) = x^2$.

For each of these, find $f'(2)$.

Solution. Let us find $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

a) $\dfrac{1-1}{\Delta x} = 0$, so $f'(x) = 0$.

b) $\dfrac{2(x+\Delta x) - 2x}{\Delta x} = \dfrac{2\Delta x}{\Delta x} = 2$. Note here that $\Delta x \neq 0$ is a very small number approaching zero, but not equal to zero, so we can perform the division. Hence $f'(x) = 2$.

c) $\dfrac{(x+\Delta x)^2 - x^2}{\Delta x} = \dfrac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} = 2x + \Delta x$. As Δx tends to zero, the sum $2x + \Delta x$ tends to $2x$ (x is independent of Δx). Thus,

$$f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

for any value of x.

At the point $x = 2$, we have a) $f'(2) = 0$; b) $f'(2) = 2$; c) $f'(2) = 4$.;

Some more examples can be found in this Maths Genie video. Also, you can study the Khan Academy unit Differentiation: definition and basic derivative rules up to Quiz 1 to get more familiar with the concept of a derivative.

The result of Example 1 can be generalized.

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1} \text{ for } n \neq 0. \tag{2}$$

This is derived for positive integer values of n in this video by MathGUIDE (this video uses the binomial expansion you are familiar with from the Assignment on combinatorics and probabilities).

Problem 2 (3 marks). Show from the first principles that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2} \tag{3}$$

and check that this result is in agreement with the power rule.

We will later show that the power rule also applies to other negative integer n's. The rule also holds for fractional exponents, but the derivation has to be deferred to later in the course when you have learned exponential and logarithmic functions.

3 Linearity of differentiation

Most of this assignment is dedicated to differentiating complex functions. To find a derivative of such a function, we "divide and conquer": express it as combinations of simpler functions — such as polynomial or trigonometric functions. Then we apply two sets of rules. One is the formulae for differentiating simple functions, such as the power rule we just studied. The second set of rules tells us how to differentiate functions that are combined in a certain way. Here is the first such rule:

$$[f(x) + g(x)]' = f'(x) + g'(x); \tag{4}$$
$$[a \times f(x)]' = a \times f'(x). \tag{5}$$

This can be written in a single line as

$$[af(x) + bg(x)]' = af'(x) + bg'(x). \tag{6}$$

The proof of Eq. (4) is very simple:

$$\begin{aligned}
[f(x) + g(x)]' &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) + g(x + \Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} + \frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\
&= \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] + \lim_{\Delta x \to 0} \left[\frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \\
&= f'(x) + g'(x)
\end{aligned}$$

(here we implicitly used the fact that the limit of the sum of two function is the sum of their limits). You can prove Eq. (5) in a similar manner independently.

We know that the derivative of a constant is zero (see Example 1(a)). It follows that, for any constant C,

$$[f(x) + C]' = f'(x). \tag{7}$$

Problem 3 (2 marks). Check that the function plotted in Problem 1 is $s(t) = \dfrac{t^3}{2} - t + 1$. Calculate the derivative of this function and plot it in the same graph as your solution to Problem 1.

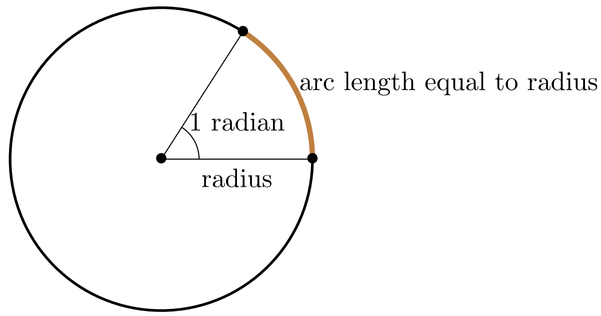
Problem 4 (2 marks). Find the tangent of the angle at which the curve $y = x^2 + x - 2$ cuts the x-axis.

4 Derivatives of trigonometric functions

4.1 Radians

To differentiate trigonometric functions, we need to introduce the *radian (rad)* — a unit of angle that is commonly used in physics and mathematics instead of degrees. The radian is defined as the angle subtended at the center of a circle by an arc that is equal in length to the radius. To determine the magnitude in radians of a given angle, we need to draw a circle (of arbitrary radius) centred at the apex of the angle and calculate

$$\text{angle in radians} = \frac{\text{length of arc inside the angle}}{\text{radius}}. \tag{8}$$



Because the circumference of a circle equals $C = 2\pi r$ we can say there are 2π radians in one circle, i.e. $360° = 2\pi$ rad. Thus 1 rad $= 360°/2\pi \approx 57.296°$. The radian feels slightly weird to begin with as we are used to a whole number of degrees making one complete circle, and 2π is not a whole number. However this is a very natural way of defining an angular unit, as it is fixed by the geometry, in contrast to the number 360 chosen for random historic reasons.

Problem 5. (1 mark) An object moves at a constant linear speed of 10 m/sec around a circle of radius 4 m. How large of a central angle does it sweep out in 3.1 seconds? (Give the answer in radians, do not use a calculator)

As evidenced by Eq. (8), the radian is a meter divided by a meter, i.e. a dimensionless unit. For this reason, it is common to omit the unit when expressing an angle in radians. For example, when we say that an angle is $\pi/2$, this implies $\pi/2$ radians, i.e. $90°$.

Problem 6 (2 marks). When answering at school, meaning to say "This angle is 180 degrees", Sophia said "This angle is 180". In response, the teacher asked her to draw the angle of magnitude 180. Help Sophia answer this question.

Example 2. Show that in a circle of radius r, the area A of the sector with central angle θ is $S = \dfrac{\theta r^2}{2}$, where θ is measured in radians.

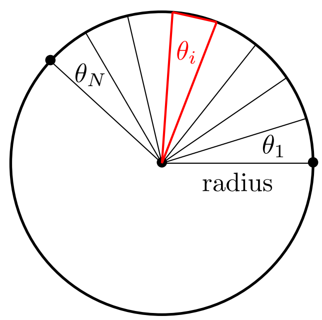
Solution. We show two methods of solving this problem. Method 1 is more straightforward, but Method 2 is more useful in solving the problem that follows.

Method 1. The ratio of the areas of the sector and the whole circle is $\dfrac{\theta}{2\pi}$. Recalling that the area of a circle is πr^2, we find for the sector

$$S = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}.$$

Method 2. We adapt the standard argument for deriving the area of a circle. Let us divide the sector into N sectors, each with a small central angle θ_i. Each of these sectors can be approximated by an isosceles triangle with the apex at the center of the circle and the base connecting the two ends of the arc. Because θ_i is small, the base is approximately the same length as the arc of that sector, i.e. $\theta_i r$. The height is approximately the radius r. Hence the area of each small sector is $S_i = \theta_i r^2/2$ and the area of the entire sector

$$S = \sum_i S_i = \sum_i \frac{\theta_i r^2}{2} = \frac{\theta r^2}{2}.$$



Problem 7 (3 marks). Show that, for an angle θ expressed in radians, the following approximation is valid:

$$\sin\theta \approx \theta,$$

where $\theta \ll 1$ rad. Check this approximation using a calculator for $\theta = 15°, 30°, 45°, 90°$.

This is an important result, which we write rigorously as

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1. \tag{9}$$

It readily follows that, for any real number A,

$$\lim_{\theta\to 0} \frac{\sin A\theta}{\theta} = \lim_{\theta\to 0} A\frac{\sin A\theta}{\theta A} = A \lim_{(\theta A)\to 0} \frac{\sin A\theta}{\theta A} = A,$$

because whenever $\theta \to 0$, also $\theta A \to 0$.

4.2 Derivatives of sine and cosine

Please study the first three sections of the Khan Academy lesson from the same unit, which present the following rules and show examples of their use.

$$\frac{\mathrm{d}\sin x}{\mathrm{d}x} = \cos x \text{ and } \frac{\mathrm{d}\cos x}{\mathrm{d}x} = -\sin x. \tag{10}$$

The fourth section in the same lesson shows derivation of these formulae. We make it optional, but we do strongly recommend that you study this section. To follow it, you will need the following identity:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \tag{11}$$

proved in this Khan Academy video.

Problem 8* (3 marks). For a real $A \neq 0$, show that

$$(\sin Ax)' = A\cos Ax; \tag{12a}$$
$$(\cos Ax)' = -A\sin Ax; \tag{12b}$$

from the first principles.

5 More rules

5.1 Product Rule

The derivative of the product of two functions can be found using the rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \text{ or simply } (fg)' = f'g + fg' \tag{13}$$

Please learn about the product rule from the corresponding video of the same Khan Academy unit.

Example 3. Find the derivative of $f(x) = \sqrt{x} \times \sin x$.

Solution: $f'(x) = (\sqrt{x})'(\sin x) + (\sqrt{x})(\sin x)' = \dfrac{1}{2\sqrt{x}} \sin x + \sqrt{x} \cos x$

Problem 9 (2 marks). Find the derivative of $f(x) = (x - 1)(x^2 + 5x - 7)$

a) using the product rule;

b) by expanding $f(x)$ into a single polynomial.

Check that the results are the same.

Problem 10 (3 marks). Find the derivatives of (a) $\sin^2 x$ and (b) $\cos^3 x$ using

- the product rule;
- Eqs. (12) and the following identities: $\cos 2x \equiv 1 - \sin^2 x$; $\cos 3x \equiv 4\cos^3 x - 3\cos x$.

Check that the results are the same.

5.2 Chain Rule

The chain rule is the most difficult of the basic rules of differentiation. This rule helps you find the derivative of a *composite function* $f(g(x))$ — that is, a function whose argument is another function $g(x)$. For example, suppose you need to find the derivative of $\sin(x^3)$. In this case, $g(x) = x^3$ and $f(g) = \sin g$.

The chain rule is as folows:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = \frac{df}{dg} \times \frac{dg}{dx} = f'_g(g) \times g'_x(x), \tag{14}$$

where $f'_g(g)$ is the derivative of f with respect to g and $g'_x(x)$ is the derivative of g with respect to x.

Please watch this MATHGuide video to learn the derivation of the chain rule.

Example 4. Find the derivatives of

a) $y = \sin(x^3)$;

b) $y = \left(3\sin x + \dfrac{8}{x}\right)^7$.

Solution.

a) We can write the function as $y = f(g) = \sin g$, where $g(x) = x^3$. Using the chain rule:

$$y' = f'_g(g) \times g'_x(x) = (\sin g)'_g \times (x^3)'_x = \cos g \times \left(3x^2\right) = 3x^2 \cos x^3.$$

b) We can write the function as $y = f(g) = g^7$, where $g(x) = 3\sin x + \dfrac{8}{x}$. Then

$$y' = f'_g(g) \times g'_x(x) = (g^7)'_g \times \left(3\sin x + \frac{8}{x}\right)'_x = 7g^6 \times \left(3\cos x - \frac{8}{x^2}\right) = 7\left(3\sin x + \frac{8}{x}\right)^6 \times \left(3\cos x - \frac{8}{x^2}\right).$$

Example 5. Derive relations (12) using the chain rule.

Solution: Let $f(g) = \sin g$ and $g(x) = Ax$. We find $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin Ax = \dfrac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'_g(g) \times g'_x(x) = \cos g \times A = A\cos Ax$. The other relation is obtained similarly.

Problem 11 (2 marks). Find the derivatives of (a) $\sin^2 x$ and (b) $\cos^3 x$ using the chain rule. Compare with the result of Problem 10.

Problem 12 (3 marks). Find the derivative of $y = (\cos x - 1)^2$

a) by expanding y into a polynomial function of $\sin x$ and using the identity $\cos 2x \equiv 2\cos^2 x - 1$;

b) using the product rule;

c) using the chain rule.

Check that the results are the same.

Problem 13 (2 marks). Prove the power rule for an arbitrary negative integer n.

5.3 Quotient Rule

The last rule for differentiation that you will need is the *quotient rule*:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad \text{or simply} \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{15}$$

The derivation of the quotient rule can be found in this Khan Academy video.

Combining the above rules of differentiation allows you to find derivatives of most functions, however complex.

Problem 14 (4 marks). Find the derivatives of the following functions: a) (PAT 2016) $x \sin x^2$; b) $x/(a-x)^2$; c) $x/(a-x)^{1/2}$; d) $(x^2 - 4x + 5)/(3x^2 + 2x - 7)$. Make sure to simplify your answer.

6 Appendix