Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 12

Mathematics Assignment 01

Coordinate Geometry and Vectors

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Due 10 October 2024

This is the first Mathematics assignment from COMPOS Y12. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in tutorials and webinars, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 47 marks

1 Theoretical Recap

We shall start by compiling a few fundamental results (covered in year 10) on co-ordinate geometry and vectors. You should be already familiar with them, but we provide links to Art of Problem Solving and Khan Academy videos in case you need a reminder. We will go through some of these formulae again in subsequent sections.

1.1 Coordinate Geometry

- The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

  How to calculate the distance between two points.

- The midpoint (x, y) of a segment with ends $A(x_1, y_1)$ and $B(x_2, y_2)$

$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2}.$$

  How to calculate the midpoint of a segment.

- The equation of a straight line

$$y = mx + c,$$

  where m is the gradient, and $(0, c)$ is the y-intercept.
  How to write the equation of a line in slope-intercept form

- If a straight line makes an angle θ with the positive x-axis (measured anticlockwise from the axis) then the gradient of that line is
$$m = \tan\theta$$
- Equation of a line with gradient m passing through point $A(x_0, y_0)$
$$y - y_0 = m(x - x_0).$$
How to write the equation of a line in point-slope form
- Lines parallel to the x-axis are written as $y = a$, where a is a constant.
Lines parallel to the y-axis are written as $x = b$, where b is a constant.
- Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if and only if $m_1 = m_2$.
Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if and only if $m_1m_2 = -1$.
Equations of parallel and perpendicular lines.
- Equation of a circle with centre at (x_0, y_0) and radius R
$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$
How to write the equation of a circle in standard form.

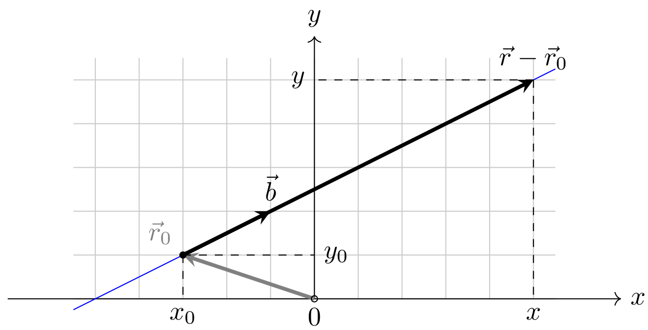
1.2 Vectors

- The product of a vector with the coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ by a scalar k is a vector
$$k\vec{a} = \begin{pmatrix} ka_x \\ ka_y \end{pmatrix}. \quad (1)$$
When a vector $\vec{a}$ is multiplied by a scalar k, the resulting vector $k\vec{a}$ is *collinear* to $\vec{a}$. If $k < 0$ the vector $k\vec{a}$ is *antiparallel* to vector $\vec{a}$.
Watch the Khan Academy video: Multiplying a vector by a scalar
- For two vectors with the coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, their sum is the vector given by
$$\vec{a} + \vec{b} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}. \quad (2)$$
Please watch the following Khan Academy videos: Adding & Subtracting vectors, Parallelogram rule and Subtracting vectors with parallelogram rule. Review an Isaac Physics example on Describing and adding vectors.
- The scalar (dot) product has two equivalent definitions.
Definition 1. For two vectors with the known coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, the dot product is the number given by
$$\vec{a} \cdot \vec{b} = a_xb_x + a_yb_y. \quad (3)$$
Definition 2. For two vectors $\vec{a}$ and $\vec{b}$ with known magnitudes and directions, the dot product is the number given by
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta, \quad (4)$$
where θ is the angle between vectors.
The equivalence of the two definitions is derived from the law of cosines, as shown in this video by Virtually Passed.

2 Straight lines

2.1 Straight lines and vectors

It is often convenient to formulate coordinate geometry in terms of vectors. In the Physics Assignment on vectors, you became familiar with the notion of the radius-vector $\vec{r}_A$ of a point A, which is the vector from the origin to that point. The coordinates $\begin{pmatrix} x_A \\ y_A \end{pmatrix}$ of the radius vector are the same as the coordinates of the point. The coordinates of the vector connecting two points A and B are then $\vec{r}_B - \vec{r}_A = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$.



A straight line passing through a point with the radius-vector $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ consists of all points $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ such all vectors from $\vec{r}_0$ to $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ have the same direction. In other words, all these vectors are multiple of some *direction vector* $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$. Hence we can write

$$\vec{r} - \vec{r}_0 = \lambda \vec{b} \;\Rightarrow\; \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \lambda \begin{pmatrix} b_x \\ b_y \end{pmatrix},$$

where λ is a number (a scalar). We can rewrite this as a pair of equations

$$\begin{aligned} x - x_0 &= \lambda b_x; \\ y - y_0 &= \lambda b_y. \end{aligned} \tag{5}$$

Dividing both sides of these equations by each other, we obtain the familiar equation of a straight line (in the point-slope form):

$$\frac{y - y_0}{x - x_0} = m,$$

where $m = b_y/b_x$ is the gradient (slope) of the line. We can rewrite the latter equation as

$$\frac{x - x_0}{b_x} = \frac{y - y_0}{b_y}. \tag{6}$$

Parallel lines have direction vectors that are the same (or proportional to each other), meaning that they have the same slope. Perpendicular lines have perpendicular direction vectors. To relate the coordinates of these vectors, let us recall the two definitions of the scalar product of two vectors

$$\vec{b}_1 \cdot \vec{b}_2 = |b_1||b_2| \cos\theta = b_{1x} b_{2x} + b_{1y} b_{2y},$$

where θ is the angle between them. For $\vec{b}_1 \perp \vec{b}_2$, $\theta = \pi/2$ and $\cos\theta = 0$. Hence

$$b_{1x}b_{2x} + b_{1y}b_{2y} = 0 \;\Rightarrow\; \frac{b_{1x}}{b_{1y}} = -\frac{b_{2y}}{b_{2x}} \;\Rightarrow\; m_1 = -\frac{1}{m_2},$$

which is again a familiar result.

Example 1. **(PAT 2010)** Find the equation of the line passing through the points $A(2,3)$ and $B(1,5)$ in the xy plane.

Solution. To find the equation of the line passing through points $A(2,3)$ and $B(1,5)$ calculate the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{1-2} = -2.$$

Now we can write the equation using the gradient and the coordinates of one of the points, for example $A(2,3)$:

$$y - 3 = -2(x-2).$$

Rearranging the terms we obtain $y = -2x + 7$.

Problem 1 (2 marks). Points $A(0,0)$, $B(6,0)$ and $C(4,4)$ are vertices of a triangle. Write the equations of the three medians[1] of that triangle, check that they intersect at the same point and find the coordinate of that point. Check that this point divides each median in a ratio of 2:1.

Problem 2 (4 marks). Points $A(-4,-2)$, $B(-3,1)$ and $C(-1,-5)$ are corners of an isosceles trapezium $ABCD$. Find the coordinates of D if AB is parallel to DC.

Problem 3. (3 marks) $ABCD$ is a rhombus. M and N are the midpoints of BC and CD respectively. Find $\angle MAN$, if $\angle ADC = 120°$ and $AB = 12$.

Problem 4. (5 marks) $ABCD$ is a parallelogram. $AB = 3\sqrt{2}$, $AD = 8$, $\angle BAD = 45°$, M is a point on CD, $CM : MD = 1:2$. N is a point on AD, $AN : ND = 3:1$. Find:

a) the length of AM;

b) the length of BN;

c) the acute angle between AM and BN.

Problem 5. (4 marks) ABC is an equilateral triangle with side 12. M is a point on BC such that the ratio of the areas of ΔABM and $\Delta ACM = 1:5$. Find:

a) length of AM;

b) the angle $\angle AMB$.

Problem 6*. (4 marks) Find the length of the angle bisector AM of the triangle ABC, if $AB = c$, $AC = b$ and $\angle A = \alpha$.

[1] A *median* of a triangle is the line connecting one of its vertices to the midpoint of the opposite edge. It is known from Euclidean geometry that the three medians of any triangle intersect at the same point and that this point (the triangle's *centre of mass*) divides each median in a ratio of 2:1. Please watch the video by MIT OpenCourseWare for a proof.

3 Circles

3.1 Introductory problems

Example 2 (PAT 2013). Find the equation of the straight line that passes through the centres of the two circles:

$$x^2 + 4x + y^2 - 2y = -1$$

and

$$x^2 - 4x + y^2 - 6y = 3.$$

Solution: To find the centres of the circles we need to complete the square and write the equations for the circles in the standard form:

$$x^2 + 4x + y^2 - 2y = -1 \;\Rightarrow\; x^2 + 4x + 4 + y^2 - 2y + 1 - 5 = -1 \;\Rightarrow\; (x+2)^2 + (y-1)^2 = 4,$$

therefore the first circle has the centre at $(-2, 1)$, and for the second, similarly,

$$(x-2)^2 - 4 + (y-3)^2 - 9 = 3,$$

meaning the second circle has the centre at $(2, 3)$. All we need to do now is to write a line that passes through the points $(-2, 1)$ and $(2, 3)$, which is a standard task:

$$y - 1 = \frac{3-1}{2-(-2)}(x - (-2)),$$

which can be rearranged as

$$y = \frac{x}{2} + 2.$$

Problem 7 (1 mark). Find the equation of the straight line which is parallel to the x axis and crosses the circle $x^2 + y^2 = 9$ in points A and B in such a way that $|AB| = 2$. Consider all possible solutions.

Problem 8. (PAT 2008, 2 marks) The points $(5, 2)$ and $(-3, 8)$ are at opposite ends of the diameter of a circle. Determine the equation of the circle.

Problem 9 (PAT 2018, 3 marks). Determine the area inside the circle defined by:

$$x^2 + y^2 - 8x + 4y + 4 = 0$$

but outside the triangle bounded by the three lines below:

$$\begin{aligned} y &= x - 7 \\ y &= \frac{1}{5}(2x - 29) \\ x &= 7 \end{aligned}$$

3.2 Inscribed and circumscribed circles

The circle *inscribed* into a triangle is the circle which is tangent[2] to all three sides of a triangle. The circle *circumscribed* around a triangle passes through all three vertices of the triangle.

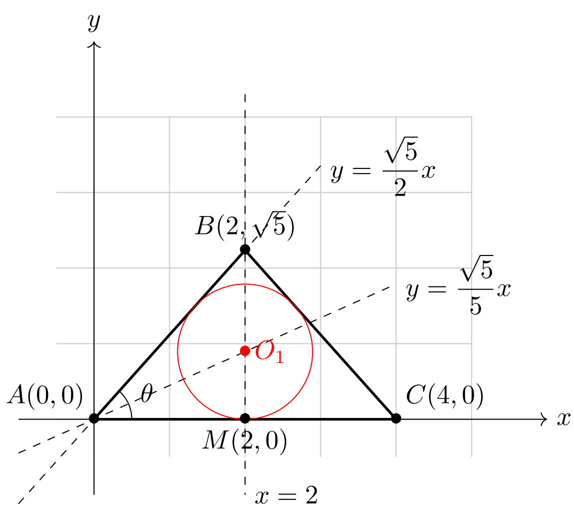
For each triangle, there exists exactly one inscribed and exactly one circumscribed circle. The centre of the inscribed circle is at the intersection of *angle bisectors* of the triangle. The centre of a circumscribed circle is at the point of intersection of the *perpendicular bisectors* of each side. Please make a drawing and convince yourself of these facts. This Krista King Math page may be helpful.

Example 3. For an isosceles triangle with sides 4, 3, 3 find the:

a) radius of the inscribed circle;

b) radius of the circumscribed circle.

Solution. We start by choosing a coordinate system and drawing a diagram.



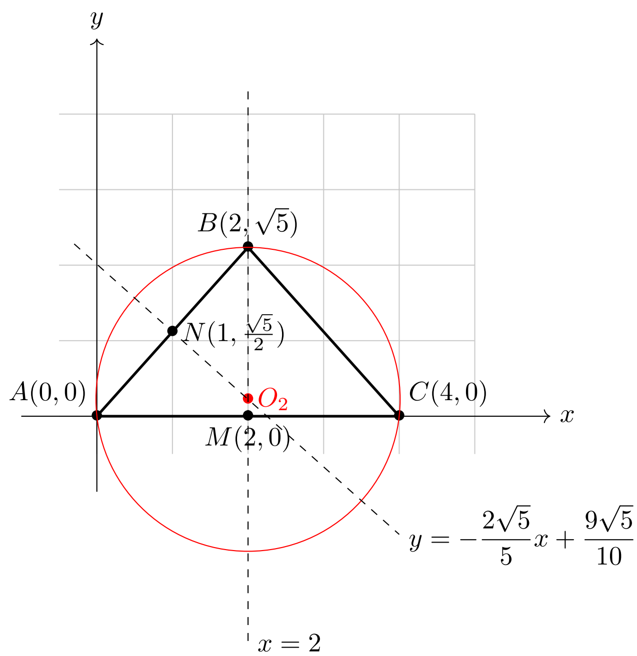
As we know, the centre of the inscribed circle is at the intersection of angle bisectors. Let us find the equations of two of the bisectors. One of them is easy: the bisector BO_1 of $\angle B$ is vertical and has equation $x = 2$.

Let us now find the bisector AO_1 of $\angle A$. We find from the Pythagoras' theorem that the height of the triangle is $\sqrt{3^2 - 2^2} = \sqrt{5}$, which gives us the y coordinate of point B. Hence $\tan\theta = \frac{\sqrt{5}}{2}$. Using the trigonometric formula for the tangent of the half-angle: $\tan(\theta/2) = \frac{\sin\theta}{1+\cos\theta} = \frac{\sqrt{5}/3}{1+2/3} = \frac{\sqrt{5}}{5}$. This is the slope of AO_1, so its equation is $y = \frac{\sqrt{5}}{5}x$.

[2]For a point P on a circle, the *tangent* line to a circle at P is a line that intersects the circle at P and has no other common points with the circle. The tangent line is always perpendicular to the radius drawn to P. Here we will show how to find the tangent lines to a given circle which pass through a given point.

Solving simultaneously with $x = 2$ we find $O_1(2, \frac{2\sqrt{5}}{5})$. The radius of the inscribed circle is equal to the length of O_1M, i.e. $r_{\text{insc}} = \frac{2\sqrt{5}}{5}$.

To find the radius of the circumscribed circle we will use the same coordinate system. The centre of the circle is O_2:



We use the same approach, but now we need the equations of two perpendicular bisectors. One of them — the perpendicular bisector of AC — is $x = 2$

The perpendicular bisect to AB has the gradient $-\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$ because it is perpendicular to AB. $N\left(1, \frac{\sqrt{5}}{2}\right)$ is the midpoint of AB. Using the point-slope formula we find the equation of line NO_2:

$$y - \frac{\sqrt{5}}{2} = -\frac{2\sqrt{5}}{5}(x - 1) \Rightarrow y = -\frac{2\sqrt{5}}{5}x + \frac{9\sqrt{5}}{10}.$$

To find the point O_2, substitute $x = 2$: $O_2\left(2, \frac{\sqrt{5}}{10}\right)$

The radius of the circumscribed circle is equal to the length of BO_2: $R_{\text{circ}} = \sqrt{5} - \frac{\sqrt{5}}{10} = \frac{9\sqrt{5}}{10}$.

Answer: $\frac{2\sqrt{5}}{5}$; $\frac{9\sqrt{5}}{10}$.

This example shows a neat property of the coordinate method: it can be used to solve problems that are not phrased in the coordinate language. Indeed, almost every geometry problem can be solved using this method (albeit not always elegantly). A few more examples will follow.

Problem 10 (3 marks). Find the equation of the circle inscribed in the triangle with vertices $A(1, 0)$, $B(5, 0)$ and $C(3, \sqrt{5})$.

Hint: You may notice a special property of the triangle that simplifies the calculation.

Problem 11 (2 marks). Write down the equation of the circle that passes through the vertices of the triangle created by line $2x + 3y = 6$, x-axis and y-axis.

4 Coordinate method in 3D-geometry

In 3D, we have three coordinate axes, which are mutually orthogonal. Every point is defined by its coordinates (x, y, z) by orthogonal projections on the corresponding axis (just as we have it in a two-dimensional case). We can also define a three-dimensional vector. For two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ the vector from A to B is $\overrightarrow{AB} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$.

Some facts from 3D-geometry are direct generalisation of their 2D-counterparts:.

- The distance between two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ (the magnitude of the vector $\overrightarrow{AB}$) is

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- The formula for a line passing through a point $M(x_0, y_0, z_0)$ along a vector (a, b, c) is $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$. This formula is a generalisation of Eq. (6) and further explained in the How To Find The Vector Equation of a Line video by The Organic Chemistry Tutor.
- The two familiar definitions of the scalar product of two vectors are directly extended from 2D to 3D:

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta = a_x b_x + a_y b_y + a_z b_z,$$

  where θ is an angle between vectors $\vec{a}$ and $\vec{b}$. Hence the angle between two vectors can be found via

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}.$$

- The angle ϕ between the two planes

$$\alpha : a_1 x + b_1 y + c_1 z + d_1 = 0$$
$$\beta : a_2 x + b_2 y + c_2 z + d_2 = 0$$

  is given by

$$\cos\phi = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

  Please derive this result on your own, observing that ϕ is equal to the angle between the normal vectors to the two planes in question.

Here are a few definitions of 3D shapes that you will need to solve the problems below.

- A *polyhedron (polytope)* is a shape whose surface consists of a finite number of polygons (called the *faces* of the polyhedron). A polyhedron is *regular* if its faces are congruent regular polygons.
- A *prism* is a polyhedron that consists of two congruent polygons (known as the prism's *bases*) in two parallel planes as well as segments connecting their corresponding vertices. A prism is *right* if these segments are perpendicular to the bases. A right prism is *regular* if its bases are regular polygons. The *height* of a prism is the distance between the bases. A *cylinder* is an analog of a prism with circles as the bases.
- A *pyramid* is a polyhedron that consists of a polygon (known as the pyramid's *base*), a point outside the plane of that polygon (the *apex*) and segments connecting the vertices of the base to the apex. The *height* of a pyramid is the distance between the apex and the plane of the base. A pyramid is *regular* if its base is a regular polygon and its lateral edges are all equal in length[3]. A *cone* is an analog of a pyramid with a circle as the base.
- A *sphere* is the set of all points in space that are located at a certain distance (the *radius*) from a given point (the *centre*). We call a sphere *circumscribed* around a polyhedron if all the vertices of the polyhedron lie on the sphere. We call a sphere *inscribed* in a polyhedron if it touches all its faces.

Five Platonic Solids. Since ancient Greece, it has been known that there exist just five regular polyhedrons, also known as the five Platonic solids: tetrahedron, cube (hexahedron), octahedron, dodecahedron and icosahedron. We invite you to watch this video by Zach Star which explains why there are only five of them.

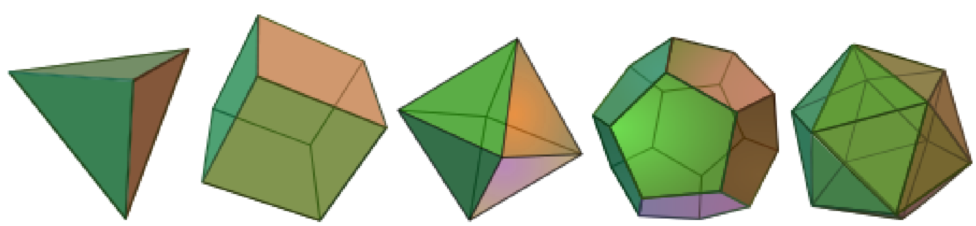


Image source: Wikipedia.

Example 4. In a regular tetrahedron with edge length 1, find

a) the height,
b) the radius R of the circumscribed sphere.

Solution. Let $\triangle ABC$ be the base of the tetrahedron, and M the midpoint of BC. Because the median AM is also the height of $\triangle ABC$ (an equilateral triangle with side length 1), we find $AM = 1 \times \cos 30^\circ = \frac{\sqrt{3}}{2}$. The center of mass O of $\triangle ABC$ divides AM in proportion $AO : OM = 2 : 1$, hence $AO = \frac{1}{\sqrt{3}}$ and $OM = \frac{1}{2\sqrt{3}}$.

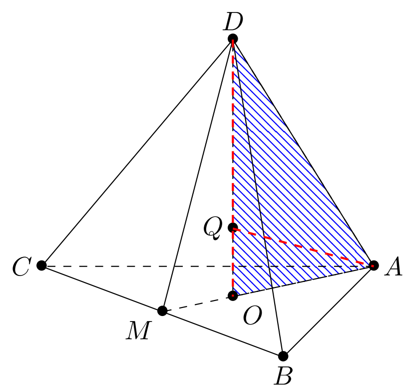
The base center O is the orthogonal projection of the regular tetrahedron's apex D onto the base plane. Hence AOD is a right-angled triangle. Its leg $DO = \sqrt{AD^2 - AO^2} = \sqrt{\frac{2}{3}}$ is the height of the tetrahedron.

The centre Q of the circumscribed sphere is located on line DO, and we must have $QA = QD = R$. In the right triangle AOQ, one cathetus $OQ = DO - QD = \sqrt{\frac{2}{3}} - R$, the other cathetus $AO = \frac{1}{\sqrt{3}}$ and the

[3] A regular prism or a regular pyramid are not necessarily regular poyhedrons according to these definitions.

hypothenuse $AQ = R$. Writing the Pythagorean theorem,

$$\frac{1}{3} + \left(\sqrt{\frac{2}{3}} - R\right)^2 = R^2 \;\Rightarrow\; 1 - 2R\sqrt{\frac{2}{3}} + R^2 = R^2 \;\Rightarrow\; R = \frac{\sqrt{3}}{2\sqrt{2}}.$$



Problem 12 (3 marks). For a regular tetrahedron with edge length 1, find

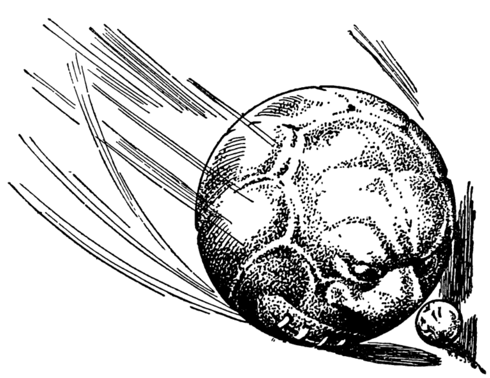
a) the angles the lateral edges make with the base;

b) the angles the lateral faces make with the base[4].

Problem 13 (4 marks). Find the angle between the adjacent faces of a regular octahedron.

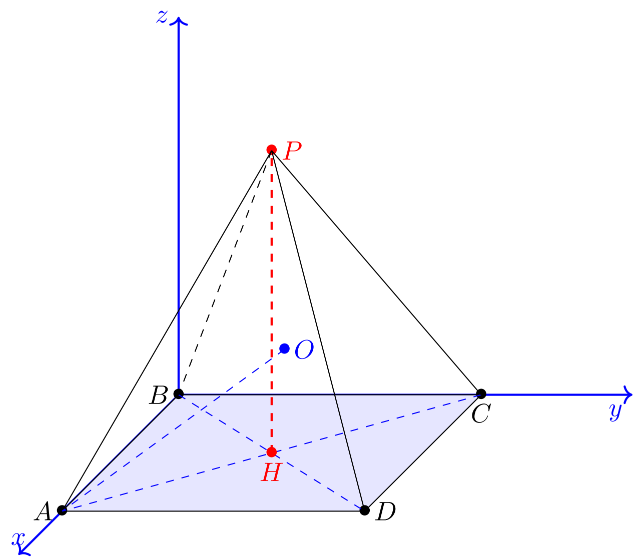
Problem 14 (4 marks). A football is placed in the corner of a rectangular room, touching two walls and the floor. A smaller ball (e.g. a tennis ball) is placed in the same corner, touching both walls, the floor and the football. Find the ratio of the radii of the two balls.

Hint: introduce a 3D coordinate system with the origin in the corner of the room.



[4]The angle between two planes is equal to the angle between the normal (perpendicular) vectors of the two planes.

Example 5. PH is the height of a regular square pyramid $PABCD$, O is the intersection point of medians of triangle BCP. Find the angle between the lines PH and AO, given that AB $= PH$.



Solution. Let us introduce a coordinate system as shown in the diagram and determine the coordinates of points A, O, P, and H. Suppose the side of the base equals 1, then we will obtain the coordinates $A(1,0,0)$, $H(1/2,1/2,0)$, $P(1/2,1/2,1)$. To find the coordinate of O, we write the median of $\triangle BCP$ drawn from point B is as a vector $\frac{1}{2}\left(\vec{BP}+\vec{BC}\right)$. Because the median intersection point divides that vector in the ratio of 2:1, we can write $\vec{BO} = \frac{1}{3}\left(\vec{BP}+\vec{BC}\right)$ and hence the coordinates of O are $(1/6, 1/2, 1/3)$.

Then we get the vectors $\vec{AO} = \left(-\frac{5}{6}, \frac{1}{2}, \frac{1}{3}\right)$, $\vec{HP} = (0,0,1)$. And so $\cos\phi = \frac{\frac{1}{3}}{\sqrt{1}\times\sqrt{\frac{38}{36}}} = \frac{2}{\sqrt{38}} \Rightarrow \phi = \arccos\frac{2}{\sqrt{38}}$.

Problem 15 (3 marks). In a regular square pyramid $SABCD$ with apex S, the height is equal to the diagonal of the base. Point F lies at the midpoint of edge SA. Find the angle between lines SD and BF.