Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 12

Mathematics Assignment 02

Trigonometric Transformations and Equations

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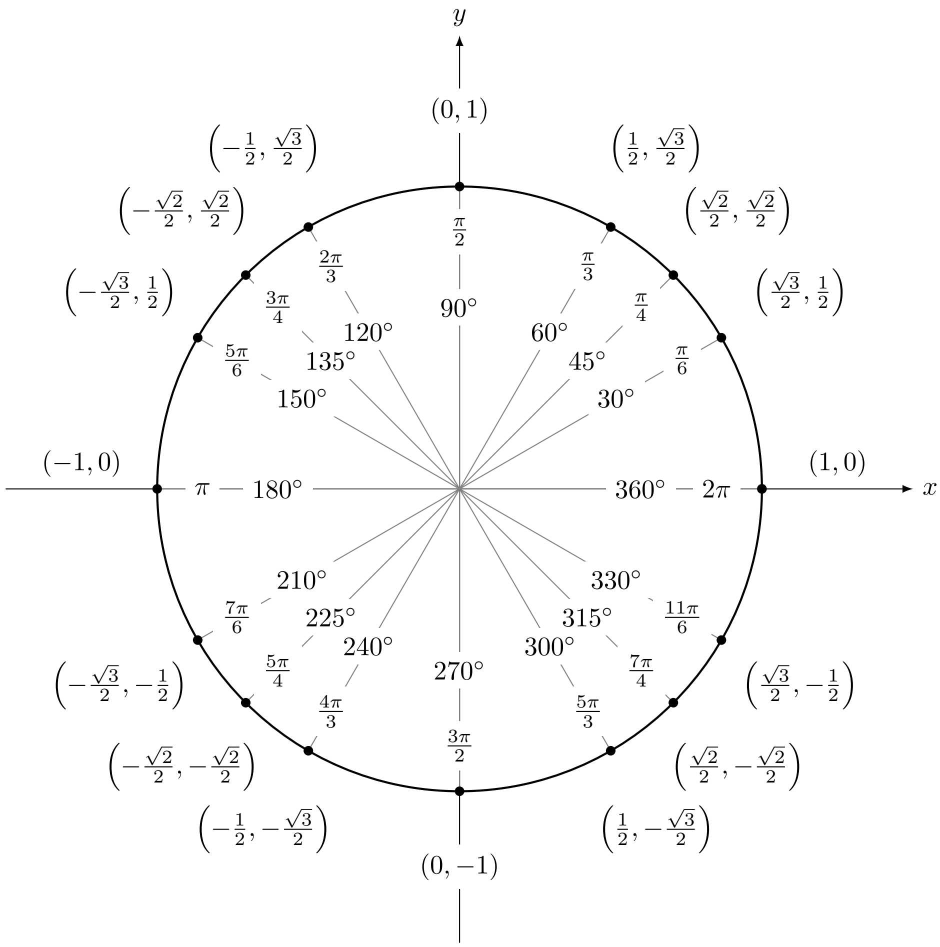
This is the second Mathematics assignment from COMPOS Y12. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in tutorials and webinars, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 49 marks.

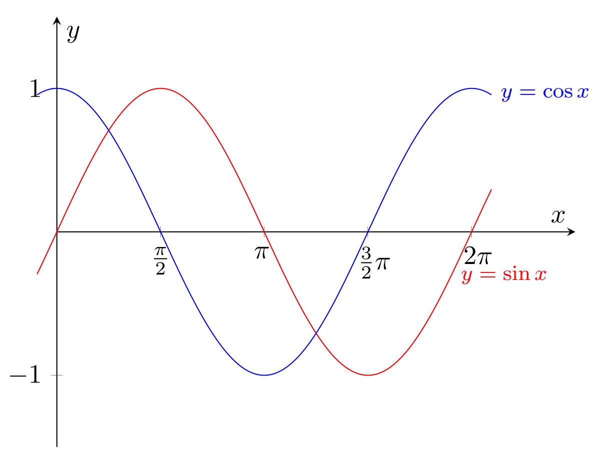
1 Trigonometric Functions

Familiarize yourself with the concept of *the unit circle* with the help of this Khan Academy video.



The coordinates of any point on the unit circle are $(x, y) = (\cos\theta, \sin\theta)$. The Pythagorean theorem gives rise to the *main trigonometric identity* $\sin^2 x + \cos^2 x = 1$.

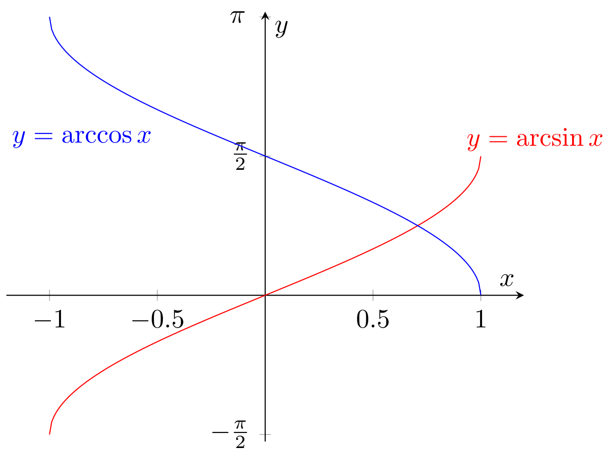
The main trigonometric functions $y = \sin x$ and $y = \cos x$ functions are shown on the graph. The *domain* of both functions is $x \in \mathbb{R}$ (the set of all real numbers) and the *range* of both functions is $-1 \leq y \leq 1$.



Here are the two most important *inverse trigonometric functions* $y = \arcsin x$ and $y = \arccos x$. They are also commonly known as the *inverse sine* and *inverse cosine*: $y = \sin^{-1} x$ and $y = \cos^{-1} x$ respectively. These functions solve the following equations in terms of y: $x = \sin y$ and $x = \cos y$, respectively.

Because sine and cosine are periodic functions[1], the above equations have multiple solutions in terms of y. On the other hand, any mathematical function $y(x)$ must have *only one* y corresponding to a given x. To address this requirement, we always look for the solution of $x = \sin y$ in the interval $y \in [-\pi/2, \pi/2]$ and $x = \cos y$ in the interval $y \in [0, \pi]$. In other words, the *range* of $\arcsin x$ is $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ and the *range* of $\arccos x$ is $0 \le y \le \pi$. The *domain* of both functions is, of course, $[-1, 1]$ (i.e. $-1 \le x \le 1$).

Not surprisingly, the graphic representations of the direct and inverse functions are obtained by exchanging the x and y axes (reflecting the graphs of $y = \sin x$ and $y = \cos x$ over the line $y = x$), keeping in mind the range restriction of the inverse functions as discussed above.



Obviously, $\sin(\arcsin x) = x$ and $\cos(\arccos x) = x$, where $-1 \le x \le 1$. The reverse is slightly more complicated: $\arcsin(\sin x) = x'$ where x' is obtained by adding or subtracting 2π ($360°$) to x or $\pi - x$ ($180° - x$) until the result is within the range $[-\frac{\pi}{2}; \frac{\pi}{2}]$ ($[-90°; 90°]$). Accordingly $\arccos(\cos x) = x'$, where x' is obtained by adding or subtracting 2π ($360°$) to x or $-x$ until the result is within the range of $[0; \pi]$ ($[0; 180°]$).

To illustrate this:

$\sin(\arcsin 0.2) = 0.2$; $\arcsin(\sin 420°) = 420° - 360° = 60°$; $\arccos\left(\cos \dfrac{-19\pi}{6}\right) = \dfrac{-19\pi}{6} + 4\pi = \dfrac{5\pi}{6}$. Here we subtracted $360°$ because $420°$ is outside the range $-90° \le x \le 90°$, and added 4π because $\dfrac{-19\pi}{6}$ is outside the range $0 \le x \le \pi$.

$\arcsin(\sin 120°) = 180° - 120° = 60°$. Here we kept in mind that there are two points on the unit circle, corresponding to one sine value. We choose the one on the right half of the circle as it corresponds to the range $[-90°; 90°]$.

Example 1. Find the following: a) $\arcsin\left(\sin \dfrac{7\pi}{3}\right)$; b) $\sin(\arccos 0.6)$.

[1]A function $f(x)$ is called *periodic* with period T if $f(x + T) = f(x)$ for any x.

Solution.

a) $\arcsin\left(\sin\dfrac{7\pi}{3}\right) = \dfrac{7\pi}{3} - 2\pi = \dfrac{\pi}{3}$.

b) Using the main trigonometric identity we can write

$$\sin(\arccos 0.36) = \pm\sqrt{1-\cos^2(\arccos 0.6)} = \pm\sqrt{1-0.36} = \pm 0.8.$$

We note that $\arccos 0.6$ is an angle in the first quadrant, therefore its sine cannot be negative.

Answer: 0.8.

Problem 1 (6 marks). Sketch the graphs of the following functions clearly indicating the range and domain:

a) $y = \sin(\arcsin x)$;

b) $y = \arcsin(\sin x)$;

c) $y = \arccos(\cos x)$;

d) $y = \sin(\arccos x)$.

Problem 2 (3 marks). Solve the equation $\arcsin(3a^2 - 4a - 1) = \arcsin(a+1)$.

Further important trigonometric functions are

$$\tan x \equiv \frac{\sin x}{\cos x} \text{ and } \cot x \equiv \frac{\cos x}{\sin x}. \tag{1}$$

You can learn about these functions with the help of these Khan Academy video and MySecretMathTutor video.

Problem 3 (3 marks).

a) What are the domains and ranges of $\tan x$ and $\cot x$?

b) What are the domains and ranges of the functions $y = \arctan x$ and $y = \operatorname{arccot} x$ (the ranges must be continuous and include $y = 0$ for $\arctan x$ and $y = \pi/2$ for $\operatorname{arccot} x$)? Plot these functions. ($\arctan x \equiv \tan^{-1} x$ and $\operatorname{arccot} x \equiv \cot^{-1} x$)

Example 2. Find $\dfrac{2\sin\theta + 3\cos\theta}{3\sin\theta - 5\cos\theta}$, if $\tan\theta = 3$.

Solution. Because $\tan\theta$ is a finite real number, $\cos\theta \neq 0$, so we can divide both the numerator and the denominator by $\cos\theta$:

$$\frac{2\dfrac{\sin\theta}{\cos\theta} + 3}{3\dfrac{\sin\theta}{\cos\theta} - 5} = \frac{2\tan\theta + 3}{3\tan\theta - 5} = \frac{6+3}{9-5} = \frac{9}{4}.$$

As you can see, the answer does not contain any trigonometric expressions. This is a common feature of problems of this kind: while one might be tempted to write the answer as $\dfrac{2\sin(\arctan 3) + 3\cos(\arctan 3)}{3\sin(\arctan 3) - 5\cos(\arctan 3)}$, it would not be acceptable, because it is much more complicated than $9/4$. You should always try to simplify your answer as much as possible.

Problem 4 (PAT 2009, 2 marks). If $x = \sin t$ and $y = \tan t$, express y in terms of x.

Problem 5 (PAT 2006, 3 marks). Simplify $13\sin\left[\arctan\left(\dfrac{12}{5}\right)\right]$.
Hint (not provided in the test): Draw the corresponding right-angled triangle.

2 Trigonometric identities

When working with trigonometric expressions it is important to know the main *trigonometric identities*. An especially important pair of identities is as follows:[2]

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y; \tag{2a}$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y. \tag{2b}$$

Khan Academy provides the videos proving these sine and cosine angle addition identities.

[2] The $\pm$ and $\mp$ signs in the two sides of the equation are meant to be consistent. For example, Eq. (2b) implies two equations: $\cos(x+y) = \cos x \cos y - \sin x \sin y$ and $\cos(x-y) = \cos x \cos y + \sin x \sin y$.

Problem 6 (7 marks). Using the identities (2), prove the following.

a) $\cos(\pi/2 - x) = \sin x$; $\cot(\pi/2 - x) = \tan x$; $\sin(\pi + x) = -\sin x$;

b) $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

c) $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$; $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$;

d) $\sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}$; $\cos^2 \frac{x}{2} = \frac{1 + \cos x}{2}$; $\tan \frac{x}{2} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$;

e) $2 \cos\theta \cos\varphi = \cos(\theta - \varphi) + \cos(\theta + \varphi)$;

f) $\sin\theta \pm \sin\varphi = 2 \sin\left(\frac{\theta \pm \varphi}{2}\right) \cos\left(\frac{\theta \mp \varphi}{2}\right)$.

Note: it is allowed to use the previously proven results: for example; you can use the result of part (c) in solving part (d).

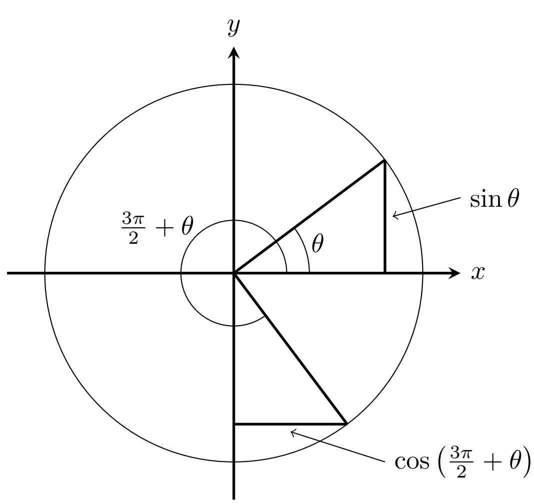
We are sorry to overload you with such a bulky problem, but these identities are of major importance for anyone doing mathematics or physics professionally. We recommend that you memorize Eqs. (2) as well as parts (c) and (d) of Problem 6. Other identities are not so important to know by heart, but you should remember that they exist, how to derive them and be able to do so quickly when necessary.

These identities, and a few additional ones, can be found on this page from Clark University. A much wider list can be found on this Wikipedia page.

Problem 7 (2 marks). Find $\sin 2\theta$ if $\sin\theta - \cos\theta = a$.

Problem 8 (2 marks). Express $y = \sqrt{3} \sin x + \cos x$ as $y = r \sin(x + \alpha)$, where $0° < \alpha < 90°$. Hence plot $y(x)$.

We also note that the derivation of the identities in Problem 6(a) — or, more generally, expressing a trigonometric function of an angle $m\frac{\pi}{2} + x$ (where m is an integer) as a function of x — does not require Eqs. (2). It suffices to simply draw (or imagine) the unit circle. Again, this is something you should be able to do quickly. For example, the drawing below shows that $\cos(3\pi/2 + \theta) = \sin\theta$.



A further important consequence of the relations in Problem 6(a) is that the plots of the functions $\sin x$ and $\cos x$, as well as $\tan x$ and $\cot x$ are symmetric to each other with respect to the vertical line $y = \pi/4$. Please convince yourself of this by reviewing Problem 3.

Example 3. Find $\cos(\alpha - \beta)$ if $\cos\alpha = \frac{1}{4}$, $\cos\beta = \frac{3}{7}$ and α, β are angles in the 1st quadrant.

Solution. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. To find $\sin\alpha$ and $\sin\beta$ we use the main trigonometric identity: $\sin\alpha = \pm\sqrt{1 - \cos^2\alpha} = \pm\sqrt{1 - \dfrac{1}{16}} = \pm\dfrac{\sqrt{15}}{4}$, and since α is in the 1st quadrant, we choose the positive sign. Similarly, $\sin\beta = \dfrac{\sqrt{40}}{7}$. Combining, $\cos(\alpha - \beta) = \dfrac{1}{4} \times \dfrac{3}{7} + \dfrac{\sqrt{15}}{4} \times \dfrac{\sqrt{40}}{7} = \dfrac{3 + 10\sqrt{6}}{28}$.

Problem 9 (2 mark). Express $\sin 301°,\ \cos 255°,\ \tan 110°,\ \cos 2\pi/3,\ \cot(-2\pi/3)$ in terms of trigonometric functions of a single argument in the range $[0, \pi/4] = [0, 45°]$.

Problem 10 (4 marks). Simplify

$$\frac{\tan 615° - \tan 555°}{\tan 795° + \tan 735°}.$$

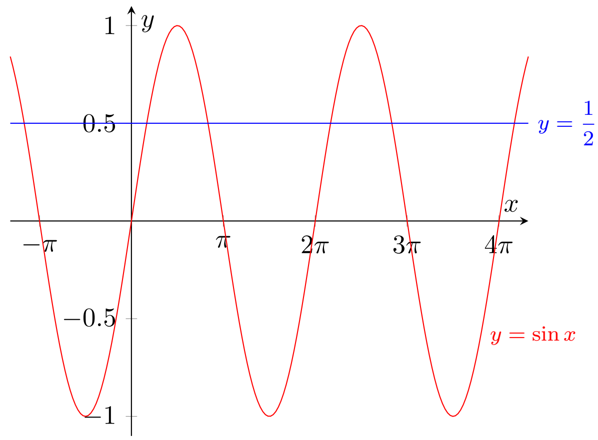
3 Trigonometric Equations

The simplest trigonometric equations are of the form

$$\sin x = a,\ \cos x = a,\ \text{or}\ \tan x = a. \tag{3}$$

When doing GCSEs and A-levels, you are usually asked to find the roots of such an equation within a given interval, in which case there is a finite number of solutions. In general, however, a trigonometric equation has an infinite number of roots. This is because trigonometric functions are periodic: $\sin x$ and $\cos x$ have a period of 2π whereas $\tan x$ and $\cot x$ have a period of π. Convince yourself of these facts by perusing the corresponding plots.

Consider, for example, the equation $\sin x = \frac{1}{2}$. If you look at the graphs of $y = \sin x$ and $y = \dfrac{1}{2}$, you will see that they have an infinite number of intersections, hence the equation $\sin x = \dfrac{1}{2}$ has infinitely many solutions.



To find them, we first identify the two solutions in the range $[0, 2\pi)$: $x = \pi/6$ and $x = 5\pi/6$. Then we use the periodicity to write the general solution

$$x = \frac{\pi}{6} + 2\pi n \text{ and } x = \frac{5\pi}{6} + 2\pi n, \tag{4}$$

where $n \in \mathbb{Z}$, i.e. n is an arbitrary integer.

This result can be written as a single expression if we use the following trick. The roots (4) can be written as $\{\ldots, -2\pi + \pi/6, -\pi - \pi/6, +\pi/6, \pi - \pi/6, 2\pi + \pi/6, 3\pi - \pi/6, \ldots\}$ — in other words, a sequence of $n\pi$ with $\pi/6$ alternately added or subtracted. This alternation can be expressed as $(-1)^n$, because $(-1)^0 = 1$, $(-1)^1 = -1$, $(-1)^2 = 1$, etc. — so that

$$x = (-1)^n \frac{\pi}{6} + \pi n. \tag{5}$$

The general solutions to simple trigonometric equations can be summarized as follows:

- $\sin\theta = a$. If $|a| > 1$ there are no solutions. If $|a| \le 1$, then $\theta = (-1)^n \arcsin a + \pi n, n \in \mathbb{Z}$;
- $\cos\theta = a$. If $|a| > 1$ there are no solutions. If $|a| \le 1$, then $\theta = \pm \arccos a + 2\pi n, n \in \mathbb{Z}$;
- $\tan\theta = a$. For any a: $\theta = \arctan a + \pi n, n \in \mathbb{Z}$.

Please convince yourself of the validity of these statements.

You will often encounter trigonometric equations that are more complex than those we just considered. They are typically solved by using some of the identities of Problem 6 to rewrite the equation in terms of a single trigonometric function (sine, cosine or tangent) and solving it with respect to that function. This reduces the equation to the form (3), which we already know how to deal with.

Example 4. Solve the equation $\sin x = \cos x$.

Solution. Dividing both sides by $\cos x$, we obtain $\tan x = 1 \Rightarrow x = \arctan 1 + \pi n, n \in \mathbb{Z} \Rightarrow x = \dfrac{\pi}{4} + \pi n, n \in \mathbb{Z}$.

Caveat: when we divide both sides of an equation by an expression, we have to make sure we are not dividing by zero. In the present case, whenever $\cos x = 0$, we cannot have $\sin x = \cos x$, so we are not losing any roots by performing the division.

Similarly, when we carelessly cancel a divider in both sides of an equation, we can illegitimately gain roots. Here is an example.

Example 5. Solve the equation $\dfrac{\sin 3x}{\sin x} = 0$.

Solution. For a fraction to be equal to zero, the numerator must equal to zero, and the denominator must not equal to zero. For the numerator, we write

$$\sin 3x = 0 \Rightarrow 3x = (-1)^k \arcsin 0 + \pi k, k \in \mathbb{Z} \Rightarrow x = \frac{\pi k}{3}, k \in \mathbb{Z} \Rightarrow x = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\pi, \pm\frac{4\pi}{3} ...$$

On the other hand, the requirement on the denominator means

$$\sin x \neq 0 \Rightarrow x \neq (-1)^n \arcsin 0 + \pi n \Rightarrow x \neq \pi n, n \in \mathbb{Z}, \text{ so } x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, ...$$

These roots must be excluded.

Answer: $x = \dfrac{\pi}{3} + \pi k$, $x = \dfrac{2\pi}{3} + \pi k$, where $k \in \mathbb{Z}$.

Problem 11 (2 marks). Solve the equation $\arccos(\cos x) = \pi - \frac{2x^2}{\pi}$.
Hint: Plot both sides of the equation.

Problem 12 (3 marks). Solve the equation $\frac{4\sin\left(\frac{3\pi}{2}+x\right)(\cos x - 1) + 3}{\sqrt{\sin x}} = 0$.

Problem 13 (PAT 2011, 2 marks). Find the values of θ between 0 and 2π which solve: $\sin\theta - 2\cos^2\theta = -1$.

Problem 14 (PAT 2020, 2 marks). Find all solutions of the following equation in the range $0 \leq \theta \leq 360°$.

$$4\cos^2\theta + 2(\sqrt{3} - 1)\sin\theta = 4 - \sqrt{3}.$$

Example 6. Solve the equation $4\cos^4\theta + 9\cos 2\theta - 1 = 0$

Solution: The second identity in Problem 6(c) says that $\cos 2\theta = 2\cos^2\theta - 1$, hence the equation becomes

$$4\cos^4\theta + 18\cos^2\theta - 10 = 0 \;\Rightarrow\; \cos^2\theta = 1/2 \text{ or } -5.$$

Only the first root is valid, meaning that $\cos\theta = \pm 1/\sqrt{2}$.

Answer: $\pi/4 + n\pi/2$, where $n \in \mathbb{Z}$.

Problem 15 (2 marks). Solve the equation $\tan 2\theta = 3\tan\theta$.

Problem 16 (4 marks). Find the values of a for which the equation $a\sin^2 x - (a-1)\sin x - 1 = 0$ has 3 distinct roots on the interval $0 \le x \le 2\pi$.

Total 49 marks

4 Extra Optional Problems

Problem 17 (modified PAT 2011). Find the values of θ between $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$ which solve:

$4\sin^3 x = \cos(x - \frac{5\pi}{2})$.

Problem 18. Find the values of θ between 0 and $\frac{3\pi}{2}$ which solve:

$2\sin\left(\frac{11\pi}{2} + x\right) \cdot \cos\left(\frac{\pi}{2} + x\right) = \sqrt{3}\cos(-2\pi - x)$.

Problem 19. Prove the following.

a) $\sin 3x = 3\sin x - 4\sin^3 x$; $\cos 3x = 4\cos^3 x - 3\cos x$;

b) $2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$; $2\sin\theta\cos\varphi = \sin(\theta+\varphi) + \sin(\theta-\varphi)$;

c) $\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$;
$\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$.