Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 13

Mathematics Assignment 01

Complex Numbers

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This is the first Y13 Mathematics assignment from COMPOS. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in tutorials and webinars, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 57 marks.

1 The Algebra of Complex Numbers

Even if you never studied complex numbers, you probably have heard of the *imaginary unit* i, which is often defined as $i = \sqrt{-1}$. This is, of course, confusing: how can one take a square root of a negative number? To clarify the confusion, let us introduce complex numbers rigorously.

A complex number z is an *ordered pair*[1] of real numbers: $z = (x, y)$, where $x, y \in \mathbb{R}$. The coefficient x is called the *real part* of z, and denoted by $\operatorname{Re} z$, while y is referred to as the *imaginary part* and denoted by $\operatorname{Im} z$. The set of all complex numbers is denoted as $\mathbb{C}$.

The fundamental arithmetical operations of addition and multiplication can be defined on complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ as follows:

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2, y_1 + y_2); \\ z_1 \times z_2 &= (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1). \end{aligned} \tag{1}$$

It is easy to check (and you should do so!) that these operations comply with the standard properties:

- *commutativity:* $z_1 + z_2 = z_2 + z_1$ and $z_1 z_2 = z_2 z_1$;
- *associativity:* $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ and $(z_1 z_2) z_3 = z_1 (z_2 z_3)$;
- *distributivity:* $(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3$.

[1] The term "ordered" means that the order of x and y matters: (x, y) and (y, x) are, generally speaking, two different complex numbers.

Moreover, we can define the complex zero $0 = (0, 0)$ and complex one[2] $1 = (1, 0)$. The salient properties of these numbers are that, for any $z \in \mathbb{C}$, we have

$$z + 0 = z \text{ and } z \times 1 = z.$$

Again, you should check that the above relations are consistent with Eqs. (1).

These properties justify the usage of the word "numbers" in the expression "complex numbers". Indeed, complex numbers as defined above can be treated just as regular numbers in terms of arithmetic operations. The set of pairs $(x, 0)$ have properties that are identical to those of real numbers; in this way, the set $\mathbb{R}$ of real numbers can be viewed as a subset of $\mathbb{C}$. To simplify the notation, we write $(x, 0)$ simply as x.

A further simplification arises if we denote the complex number $(0, 1)$ as i. As per Eq. (1), we have

$$i^2 = (0, 1) \times (0, 1) = (0 \times 0 - 1 \times 1, 0 \times 1 + 1 \times 0) = (-1, 0) = -1.$$

Moreover, an arbitrary complex number can be written as

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0) \times (0, 1) = x + yi.$$

With these rules, we can handle the addition and multiplication of complex numbers almost in the same familiar way as we handle real numbers, with the only exception that the real and imaginary parts must be treated separately. For example: $(2 + i)(3 + i) = 6 + 2i + 3i + i^2 = 6 + 5i - 1 = 5 + 5i$.

Division of complex numbers makes use of the same idea that was used to rationalise denominators involving surds. If $z = x + yi$ then we can introduce the related complex number $z^* = x - yi$ which is known as the *complex conjugate* of z. Since $zz^* = (x+yi)(x-yi) = x^2 - y^2 i^2 = x^2 + y^2$, the product of a complex number and its conjugate is always a real number. Hence $z\dfrac{z^*}{x^2 + y^2} = 1$ and thus $\dfrac{1}{z} = \dfrac{z^*}{x^2 + y^2} = \dfrac{z^*}{zz^*}$.

> When your answer is a complex number, it is appropriate to write it in the form $x + yi$, i.e. such that the real and imaginary parts can be easily identified.

Example 1.

Simplify the fraction

$$\frac{1 - i}{1 + 2i}.$$

Solution: Multiplying numerator and denominator by the complex conjugate of $1 + 2i$ gives

$$\frac{1 - i}{1 + 2i} \times \frac{1 - 2i}{1 - 2i} = \frac{1 - 2i - i + 2i^2}{1 - 4i^2} = \frac{-1 - 3i}{5}.$$

Problem 1 (1 mark). Solve for z. Give your answer in the form $z = x + iy$, where $x, y \in \mathbb{R}$.

$$(7 + i)z + 3 = 2i.$$

Problem 2 (2 marks). Solve the pair of simultaneous equations:

$$3z + 2iw = 1 + i;$$
$$3iz + 2w = 1 - i.$$

[2] Not the same as the imaginary unit, which in this notation is $i = (0, 1)$.

Problem 3 (2 marks). By writing the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where x and y are real numbers, establish the following results.

a) $(z_1 + z_2)^* = z_1^* + z_2^*$;

b) $(z_1 z_2)^* = z_1^* z_2^*$.

The introduction of complex numbers allowed mathematicians to solve equations which were previously regarded as insoluble. For example we no longer need to reject quadratic equations which have a negative discriminant. In fact, an important result due to Gauss and known as *The Fundamental Theorem of Algebra* states that any polynomial equation with complex coefficients can always be solved using complex numbers.

Example 2. Solve the cubic equation

$$x^3 - 3x^2 + 13x - 11 = 0.$$

Solution: The left-hand side factorises as $x^3 - 3x^2 + 13x - 11 \equiv (x-1)(x^2 - 2x + 11)$.

One root is hence $x = 1$. We can obtain roots for the quadratic equation $x^2 - 2x + 11 = 0$ by either using the quadratic formula or by completing the square.

$$\begin{aligned}(x-1)^2 &= -10\\ x - 1 &= \pm\sqrt{10}i\\ x &= 1 \pm \sqrt{10}i\end{aligned}$$

The roots are hence $x = 1, x = 1 + \sqrt{10}i$ and $x = 1 - \sqrt{10}i$. Here we used the fact that the equation $z^2 = -1$ has two roots: i and $-i$. We will discuss this further later on.

It is worth noting that in the last example the two non-real roots are complex conjugates of one another. One can easily show this to always be the case whenever the polynomial has real coefficients: if some complex number z is its root, so is z^*. Try doing this as an independent exercise.

Problem 4 (2 marks). Find three complex roots[3] to the cubic equation

$$z^3 + 4z^2 + z - 26 = 0.$$

Problem 5 (4 marks). Solve the quadratic equations

a) $x^2 - 2x\sin\theta + 1 = 0$ (your answer should be in terms of θ);

b) $z^2 - 2iz + 15 = 0$.

[3]Please remember that the set of real numbers is the subset of complex numbers

2 The Argand Diagram

You will all be familiar with the idea of representing the set of real numbers geometrically as the number line. It is similarly possible to associate each complex number $z = x + iy$ with the point (x, y) in the Cartesian plane. In this way of thinking of complex numbers, real numbers lie on the x-axis while imaginary numbers iy lie on the y-axis. This representation is known as the *Argand Diagram* or simply *complex plane*.

More information on Argand diagrams can be found in this Khan Academy video.

Problem 6 (6 marks). If $z = -3 + 4i$ and $w = 12 + 5i$ evaluate the following complex numbers and mark them on the Argand diagram.

a) z b) w c) z^* d) w^* e) $\frac{1}{z}$ f) $\frac{1}{2}(z + z^*)$

g) $\frac{1}{2}(w - w^*)$ h) $\frac{1}{w}$ i) zw j) z^2 k) w/z

Use several diagrams of appropriate scales to show the precise location of each answer.

3 Modulus and Argument

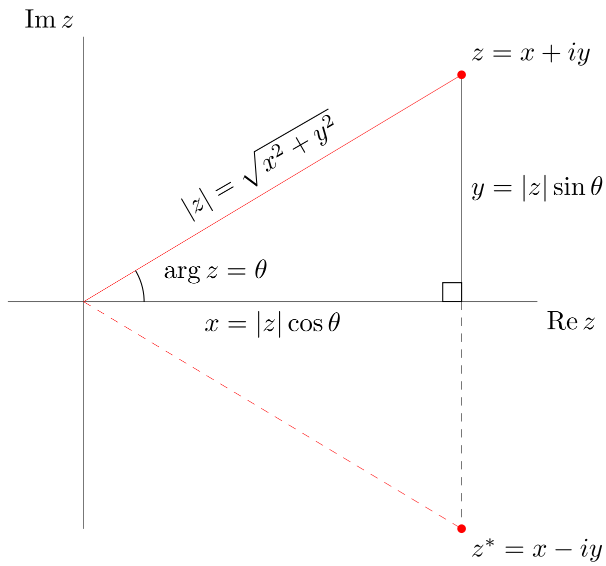
If the complex number z is marked at the point Z on the Argand Diagram then the length OZ is known as the *modulus (absolute value) of* z and is written as $|z|$. The angle that the vector OZ makes with the positive real axis, is known as the *argument of* z and is written as $\arg z$. A complex number can be specified by either its real and imaginary parts or by its modulus and argument. The two representations are related as follows:

- If we know $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$, then $|z|^2 = x^2 + y^2$ and $\tan\theta = y/x$, where $\theta = \arg z$;
- If we know $|z|$ and $\theta = \arg z$, then $x = |z|\cos\theta$ and $y = |z|\sin\theta$, i.e. $z = |z|(\cos\theta + i\sin\theta)$.

The modulus and argument form of a complex number is analogous to defining points in the plane via polar coordinates.

It is evident from the above definition that

$$|z^*| = |z| \text{ and } \arg z^* = -\arg z.$$



It is worth noting that the angle between OZ and the positive real axis is not uniquely determined as you could choose values differing by complete revolutions. By convention, the *principle value* of the argument of a complex number is always written in radians and is chosen to lie in the interval $(-\pi, \pi]$.

Problem 7 (3 marks). Mark the following numbers in the Argand Diagram:

a) $(1+i)$; b) $(1+i)^2$; c) $(1+i)^3$; $(1+i)^4$.

Give the modulus and argument in each case. Do you notice any pattern?

Problem 8 (3 marks). Find the real and imaginary parts of the complex numbers with the given values for modulus and argument.

a) $|z| = 4$, $\arg z = \frac{\pi}{3}$ b) $|z| = \sqrt{2}$, $\arg z = \frac{3\pi}{4}$ c) $|z| = \sqrt{5}$, $\arg z = \arctan 2$

The following example shows how the modulus and argument of the product, wz, is related to the modulus and argument of w and z.

Example 3. For two complex numbers v and w, prove the following relations:

$$\begin{aligned} |wv| &= |v||w|; \\ \arg(wv) &= \arg v + \arg w. \end{aligned} \tag{2}$$

Solution: We can write v and w in terms of their moduli and arguments as:

$$\begin{aligned} v &= |v| \left(\cos\theta + i \sin\theta\right); \\ w &= |w| \left(\cos\phi + i \sin\phi\right). \end{aligned}$$

Multiplying out and making use of the compound angle formulae we have:

$$\begin{aligned} vw &= |v|\,|w| \left(\cos\theta + i\sin\theta\right)\left(\cos\phi + i\sin\phi\right) \\ &= |v|\,|w| \left(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta)\right) \\ &= |v|\,|w| \left(\cos\left(\theta+\phi\right) + i\sin\left(\theta+\phi\right)\right) \end{aligned}$$

From the final line we can conclude that the modulus of vw is given by $|vw| = |v||w|$, and that $\arg v + \arg w$ is a possible choice for the argument of $\arg vw$.

Note that the principle value of $\arg(vw)$ will lie in the desired interval of $(-\pi, \pi]$ so may differ from $\arg v + \arg w$ by some integer multiple of 2π.

This critical result lies behind many interesting properties of complex numbers. In particular, it allows us to estimate and visualize on the Argand diagram the product and ratio of any two complex numbers. As an independent exercise, you can use a ruler and a protractor to check that that your answers to Problem 6 are consistent with Eqs. (2).

Problem 9 (3 marks). Use the result of Example 3 to show that

a) $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ and $\arg\left(\frac{1}{z}\right) = -\arg z$;

b) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

The result in part (b) is known as *de Moivre's Theorem*. You can find an example of how it works in Problem 7.

Problem 10 (4 marks). If $\arg z = \theta$ and $\arg w = \phi$ write down the moduli and arguments of the following complex numbers in terms of $|z|, |w|, \theta$ and ϕ:

a) z^2; b) $\frac{z^3}{w}$; c) z^*; d) $z^3 w^*$.

Problem 11 (3 marks). Let $z = 1 + i$ and $w = \sqrt{3} + i$.

a) Write down the moduli and arguments of these numbers.

b) What is the modulus and argument of wz?

c) Hence find exact expressions for $\tan 75°$ and $\cos 15°$.

4 Roots of Complex Numbers

The results in the previous section allow us to find roots of complex numbers.

Example 4. Find all complex numbers satisfying

$$z^3 + 8 = 0$$

a) By using the factor theorem.

b) By considering the modulus and argument.

Solution:

a) We can notice that $z = -2$ is a root of the equation, and hence $z + 2$ is a factor in the polynomial $z^3 + 8$. This reduces the equation to

$$(z + 2)(z^2 - 2z + 4) = 0$$

The (complex) roots of the quadratic bracket can be found using the quadratic formula as

$$z = 1 \pm \sqrt{3}i.$$

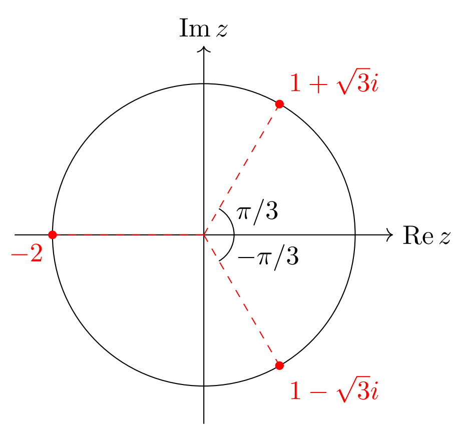
We thus have three roots as $z = -2, z = 1 + \sqrt{3}i$ and $z = 1 - \sqrt{3}i$.

b) We can express the equation as $z^3 = -8$. Our goal is to find $|z|$ and $\theta = \arg z$. According to Eqs. (2), it follows that

$$|z|^3 = |-8| = 8 \text{ and } 3\theta = \arg(-1) = \pi.$$

We are then tempted to write $|z| = 2$ and $\theta = \pi/3$.

However, this is only one of the answers for θ. We find the other answers by noticing that we can generally write $\arg(-1) = \pi + 2n\pi$ with integer n. Normally, we would neglect values of n other than 0; however, we cannot do so in this in this case because $\theta = \pi/3 + 2n\pi/3$ and different values of n now give rise to three distinguishable complex arguments: $\theta_1 = \pi/3$ for $n = 0$, $\theta_2 = \pi$ for $n = 1$ and $\theta_3 = -\pi/3$ for $n = -1$. Other values of n give answers that are equivalent to one of these and hence can be neglected.



The possible roots are therefore

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}i;$$
$$z_2 = 2\left(\cos\pi + i\sin\pi\right) = -2;$$
$$z_3 = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 1 - \sqrt{3}i.$$

We see that in complex analysis, root is a *multivalued function*: $\sqrt[n]{z}$ has n possible values distributed over a circle in the Argand diagram, the centre of which is at the origin of the complex plane and the radius is $\sqrt[n]{|z|}$.

Problem 12 (4 marks). Find all values of the following expressions:

a) $\sqrt[8]{16}$;

b) $(-27i)^{\frac{2}{3}}$;

c) $\sqrt[4]{\dfrac{-1-\sqrt{3}i}{2}}$.

Problem 13 (5 marks). Let $w \neq 1$ be a root of the equation $z^5 - 1 = 0$.

a) Show that $1 + w + w^2 + w^3 + w^4 = 0$

b) If $r = w + \frac{1}{w}$, show that $r^2 + r - 1 = 0$.

c) Solve the above equation for r. For each root, solve the equation $w + \frac{1}{w} = r$. Hence obtain and simplify all possible answers for w.

d) Solve $z^5 - 1 = 0$ using the modulus and argument method. Draw them on the Argand diagram. Use a calculator to check for consistency with the result of part (c).

e) Hence find exact expressions for $\cos\frac{\pi}{5}, \cos\frac{2\pi}{5}, \sin\frac{\pi}{5}$ and $\sin\frac{2\pi}{5}$.

5 The Complex Exponential

The exponent of the complex number $z = x + iy$ is defined as

$$e^z = e^x(\cos y + i\sin y). \tag{3}$$

In other words, the modulus of e^z is the exponential of the real part of z, while the argument of e^z equals the imaginary part of z.

For real z, the definition (3) is consistent with the standard definition. Indeed, if $y = 0$, we have $\cos y = 1$, $\sin y = 0$ and hence $e^z = e^x$.

Additionally, the definition (3) upholds the main property of exponential functions: for any two z_1 and z_2,

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

To prove this, let us write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ and define $v = e^{z_1}$ and $w = e^{z_2}$. Recalling that the modulus of e^z is the exponential of the real part of z, we write

$$|e^{z_1+z_2}| = e^{x_1+x_2} = e^{x_1}e^{x_2} = |e^{z_1}||e^{z_2}| = |v||w|.$$

Additionally, we remember that the argument of e^z equals the imaginary part of z, and hence

$$\arg e^{z_1+z_2} = y_1 + y_2 = \arg e^{z_1} + \arg e^{z_2} = \arg v + \arg w.$$

We now invoke Eqs. (2) to see that $e^{z_1+z_2}$ has the same modulus and argument as the product vw, thereby completing the proof.

The definition (3) has many interesting properties. For any real θ, we have $e^{i\theta} = \cos\theta + i\sin\theta$. Therefore de Moivre's Theorem can be formulated in the language of complex exponentials simply as

$$\left(e^{i\theta}\right)^n = e^{in\theta}.$$

The main trigonometric functions can be written as

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \text{ and } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{4}$$

or, equivalently,

$$\sin\theta = \operatorname{Im} e^{i\theta} \text{ and } \cos\theta = \operatorname{Re} e^{i\theta}. \tag{5}$$

These relations are often helpful when manipulating trigonometric functions.

Example 5. Use the complex exponent to prove the trigonometric identities

$$\sin(x+y) = \sin x\cos y + \cos x\sin y; \tag{6a}$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y. \tag{6b}$$

Solution: We know that

$$e^{i(x+y)} = \cos(x+y) + i\sin(x+y).$$

On the other hand,

$$\begin{aligned} e^{i(x+y)} &= e^x e^y \\ &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= \cos x\cos y - \sin x\sin y + i(\sin x\cos y + \cos x\sin y). \end{aligned}$$

Equating the real and imaginary parts of these expressions, we immediately obtain Eqs. 6.

Problem 14 (3 marks). Prove the identity

$$\sin^3\theta \equiv \frac{3\sin\theta - \sin 3\theta}{4}$$

using complex exponents.

Hint: Use formula (4)

Example 6.

(a) Find a simplified expression for the series

$$S = 1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots e^{i(n-1)\theta}.$$

(b) Use your expression to establish

$$1 + \cos\theta + \cos 2\theta + \dots \cos(n-1)\theta \equiv \frac{\sin\left(\frac{(2n-1)\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

(c) Obtain a similar expression for the sum

$$\sin\theta + \sin 2\theta + \dots \sin(n-1)\theta.$$

Solution:

(a) This is simply a geometric series with common ratio $e^{i\theta}$ so the sum is

$$S = \frac{\left(e^{i\theta}\right)^n - 1}{e^{i\theta} - 1}.$$

Multiplying top and bottom by $e^{\frac{-i\theta}{2}}$ gives a simpler denominator:

$$\frac{e^{\frac{(2n-1)i\theta}{2}} - e^{\frac{-i\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}} = \frac{e^{\frac{(2n-1)i\theta}{2}} - e^{\frac{-i\theta}{2}}}{2i\sin\left(\frac{\theta}{2}\right)}.$$

Multiplying top and bottom by $-i$ makes the denominator real:

$$S = \frac{ie^{\frac{-i\theta}{2}} - ie^{\frac{(2n-1)i\theta}{2}}}{2\sin\left(\frac{\theta}{2}\right)}$$

Expanding the complex exponentials, and gathering real and imaginary parts gives:

$$S = \frac{\sin\left(\frac{(2n-1)\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)} + i\left(\frac{\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{(2n-1)\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}\right).$$

b) If we expand every term of S:

$$S = 1 + e^{i\theta} + e^{2i\theta} + e^{3i\theta} + \dots e^{i(n-1)\theta} = 1 + \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta + \dots + \cos(n-1)\theta + i\sin(n-1)\theta =$$

$$= 1 + \cos\theta + \cos 2\theta + \dots + \cos(n-1)\theta + i(\sin\theta + \sin 2\theta + \dots + \sin(n-1)\theta).$$

So the sum of cosines is simply the real part of S.

c) The sum of sines is the imaginary part of S, so we have

$$\sin\theta + \sin 2\theta + \dots \sin(n-1)\theta = \frac{\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{(2n-1)\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

Problem 15 (3 marks). Show that, for $|r| < 1$,

$$1 + r\cos x + r^2\cos 2x + r^3\cos 3x + \dots = \frac{1 - r\cos x}{1 - 2r\cos x + r^2}$$

6 Complex Transformations and Locus

You can find lots of examples and extra reading material on this Khan Academy page.

The following examples show how complex numbers can be used to describe loci in the plane and to investigate transformations.

Example 7. Find and plot on the Argand diagram the set of complex numbers z which satisfy the relationship

$$|z - i| = |z - 2|. \tag{7}$$

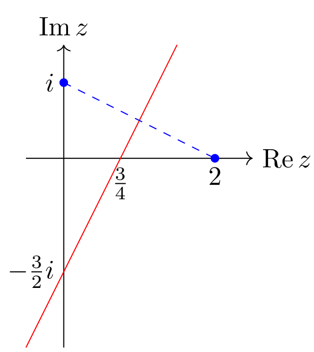
Solution: Writing z in terms of its real and imaginary parts $z = x + iy$, we have

$$|x + (y-1)i|^2 = |(x-2) + iy|^2$$

Using Pythagoras' Theorem to find the moduli, the relationship becomes:

$$\begin{aligned} x^2 + (y-1)^2 &= (x-2)^2 + y^2 \\ x^2 + y^2 - 2y + 1 &= x^2 - 4x + 4 + y^2 \\ y &= 2x - \frac{3}{2}. \end{aligned}$$

This straight line can be recognised as the perpendicular bisector of the points $(2, 0)$ and $(0, 1)$. This is not surprising: the left-hand side of Eq. (7) is the distance between the point corresponding to z and the point $(0, 1)$ corresponding to i. Similarly, the right-hand side is the distance between z and $(2, 0)$.



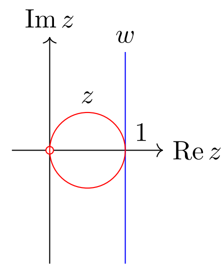
Example 8. Find the image of the line set of points $\{w : \text{Re}\,(w) = 1\}$ under the transformation

$$z = \frac{1}{w}.$$

Solution: Writing $z = x + iy$ with x, y real, and using $w = \frac{1}{z}$ we have

$$w = \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2}.$$

Since the real part of w is equal to 1 we must have that $x = x^2 + y^2$. Rearranging this as $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$, we recognise this as a circle with centre $\left(\frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$, with the point $z = 0$ excluded.



Example 9. (a) Describe the effect of the following transformations of the complex plane:

$$r(z) = e^{i\theta} z;$$
$$h(z) = z^*.$$

(b) Find an expression for complex transformation which consists of a reflection in the line passing through the origin which is inclined at an angle θ to the real-axis.

(c) Write the transformation in part (b) in terms of r_θ and h.

Solution:

(a) Since multiplying by $e^{i\theta}$ increases the argument of a complex number by θ, r performs an anti-clockwise rotation by θ about the origin.

The transformation h takes the complex conjugate of a complex number. By changing the sign of the imaginary part this consists of a reflection in the real-axis.

(b) The desired transformation can be achieved in three steps.

1) Rotate the plane by $-\theta$ about the origin (which moves the line of reflection onto the real axis). This is the transformation $z \mapsto e^{-i\theta} z$.
2) Reflect the plane in the real axis. This is the transformation $z \mapsto z^*$.
3) Rotate the plane by θ about the origin (which moves the real axis back to the line of reflection). This is the transformation $z \mapsto e^{i\theta} z$.

Hence, the transformation is

$$z \mapsto e^{i\theta} \left(e^{-i\theta} z\right)^*.$$

Since $(ab)^* = a^* b^*$ and $\left(e^{i\theta}\right)^* = e^{-i\theta}$ this simplifies to

$$z \mapsto e^{2i\theta} z^*.$$

(c) Thinking about the decomposition in part (b) the required transformation is

$$r \circ h \circ r^{-1}$$

Problem 16 (3 marks). Describe the locus in the complex plane:

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}.$$

What familiar result from Euclidean geometry does this establish?

Problem 17 (3 marks). If $\dfrac{|z-2i|}{|z+i|} = 2$ show that the locus of z is a circle. Find its centre and radius. What is the range of values of $\arg z$?

Problem 18 (3 marks). Find the image of the circle $|w| = 1$ under the transformation

$$z = \frac{w}{w-1}.$$