Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 10

Physics Assignment 01

Fluids and Hydrostatics

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Due 10th October, 2024

This is the first Physics assignment from COMPOS for Y10. The assignment goes into this topic in more detail than you have done in school. There are links to online videos which we encourage you to watch. You are free to do your own reading around this topic also.

This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. Some of the problems are hard, but don't let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Harder problems are labelled * or **.

Very similar problems will be discussed in the webinars so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 45 marks.

Hydrostatics is the area of physics which looks at how fluids behave when they are still, or *static*. For example water in a well doesn't flow or move and we can investigate the effect of the well water at different depths and on different objects using ideas from hydrostatics[1].

1 Fluids and volumes

1.1 What are fluids?

A fluid is a substance with no fixed shape which easily yields to external pressure. You will recall from the particle model that fluids (liquids and gases) have weaker bonds than solids and usually have larger spaces between their particles. This allows fluids to flow, and allows you to stick your finger into them, which is what we mean by yielding to external pressure. A solid won't do that.

Fluids take the shape of their container. We will need to consider the volume of a fluid in our

[1]When fluids move, like a rushing river or weather fronts pushing the atmosphere around, we need to use the ideas from a related area of physics called fluid *dynamics*. We won't look at fluid dynamics in this assignment.

calculations and as they take the shape of their containers, this means we need to be able to find the volumes of common 3D shapes.

1.2 Calculating volumes

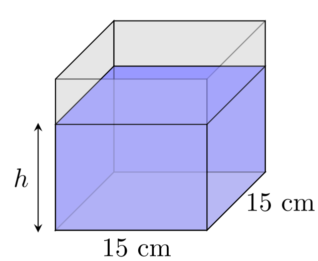
Volume in physics is usually measured in cubic metres, m^3, but it may sometimes be given in cubic centimetres cm^3, or as a capacity in millilitres (ml) or litres (l). It is useful to know how to switch between these units[2]:

- 1 ml = 1 cm^3;
- 100 cm $\times$ 100 cm $\times$ 100 cm = 1 cubic metre = 1,000,000 or 1×10^6 cubic centimetres;
- 10 cm $\times$ 10 cm $\times$ 10 cm = 1 litre = 1000ml = 1000 cm^3 = 1×10^{-3} m^3.

The volume of a liquid in a container can be found from the dimensions of the container. It is good to remember the volumes of some standard shapes.

- You will be familiar with how to find the volume of a *cuboid*: length $\times$ width $\times$ height or $V = l \times w \times h$.
- A prism is a solid shape which may have any polygon as a base and top, with a constant width from base to top. The volume of a prism is Area of base $\times$ height or $V = A \times h$. If the base of a prism is a circle, we have a cylinder. This video from Maths and Science shows a nice reminder about how to find the volume of a cylinder: $V = \pi r^2 h$.
- When you learn calculus, you will be able to derive the formula for the volume of a sphere of radius r: $V = \frac{4}{3}\pi r^3$. For now, we give it to you without proof. In this assignment you will also need the surface area of a sphere: $A = 4\pi r^2$.
- You will also need to use the formula for the area of a circle: $A = \pi r^2$ or $A = \frac{\pi d^2}{4}$.

Example 1. 1 litre of water is poured into a cube with 15 cm sides. Find how deep the water will be.



[2]Note: at COMPOS we don't always give you data you can easily convert, such as 1 ml = 1 cm^3, you can find this out yourself. Part of learning how to be good at solving science problems is knowing where to go for useful information!

Solution: We first note that a litre is the volume of a cube with 10 cm sides, so this much water does indeed fit inside the cube we're interested in. The unknown quantity is therefore the height that is reached by $V = 1000$ cm^3 of water occupying an area of $A = 15 \times 15$ cm$^2 = 225$ cm^2. We can write the equation $V = Ah$ and solve it to find the height which gives: $h = \dfrac{V}{A} = \dfrac{1000}{225}$ cm$= 4.4$ cm.

Symbolic and numeric answers The best practice when solving a physics problem is to keep your calculations symbolic (as algebra) and try to obtain the answer as an algebraic expression. If symbolic notation is not given to you in the problem, you should introduce it yourself, — like we did in the above example, introducing the letter V for volume, the area as A and the height as h. You can then substitute any numerical values into your algebraic result and calculate the answer as a number. You should keep in mind that almost every physical quantity has a unit and always show it if you state a final numerical result, like we did in the example above.

It is a *faux pas* to omit the units. Imagine you wrote that the height is equal to $h = 4.4$, your reader will be left to guess if that is 4.4 metres, nanometres or parsecs that you had in mind!

Problem 1 (2 marks). An Olympic swimming pool is 50 m long, 25 m wide and 2 m deep. Loch Ness in the Scottish highlands has a volume of 7.4×10^{12} millilitres. How many Olympic pools can it fill?

1.3 Density

Did you ever wonder why 1 liter of water weighs 1 kg? In 1790s, when the metric system was first introduced in France, the kilogram was *defined* as the mass of a liter of water at 0°C[3]. This definition relies on the *density* of water: how much mass there is for a particular volume:

$$\text{Density } (\rho) = \frac{\text{mass } (m)}{\text{volume } (V)} \text{ in units of kg} \cdot \text{m}^{-3}. \tag{1}$$

The density, indicated by the Greek letter ρ (rho), is useful because it is a property of a substance and is independent of (does not depend upon) the amount of the substance. If we take different amounts of water, measure their masses and volumes and calculate each density using Eq. (1), we will always get the same result. The density of a liquid depends on a combination of factors; the mass of its atoms or molecules, whether any substance is dissolved in it, and how tightly bound the atoms and molecules are to each other, which is why seawater is denser than distilled water.

Problem 2 (3 marks). Calculate the density of the following substances in kg·m^{-3}. Can you identify them? (Remember you can look things up yourself).

a) 1 litre of this liquid weighs 7.74 N (use $g = 9.81$N/kg);
b) 1 ml of this liquid on the pan of a balance gives the same reading in grams as 1.36 $\times 10^{-5}$m^3 of water;
c) 154.56 kg of this metal would form a cube with 20cm edges.

[3]The kilogram was later redefined a few times; the current definition (2019) is based on the atomic spectrum. However, each new definition was kept consistent with the previous one.

Problem 3* (4 marks). About 75% of the atmosphere surrounding Venus is found within 20 km of the surface, beyond this point it thins considerably with height. The radius of Venus is about 6,052 km. Estimate the mass of the atmosphere stating any assumptions you have made. The density of the atmosphere near Venus's surface is about 65 kg/m^3.

Problem 4 (4 marks). Estimate by how much the water level of the world's oceans will rise in the following situations. Oceans cover 71% of the Earth's surface.

a) If Antarctic ice melted. The mass of Antarctic ice is about 2.4×10^{19} kg, density 920 kg/m^3.

b) If all 8 billion humans on planet Earth simultaneously dived into sea. Make realistic assumptions about the dimensions of an average human body to estimate its volume.

2 Pressure and depth

2.1 Pascal's law

A fluid exerts a force on any surface[4] exposed to that fluid. This push from the fluid is a pressure due to the impact of the fluid molecules. When a molecule hits a surface, it rebounds, exerting a small force on the surface. Millions of such collisions per second give rise to a macroscopic force.

$$\text{Pressure } (P) = \frac{\text{Force } (F)}{\text{Area } (A)} \quad \text{in units of N} \cdot \text{m}^{-2} \text{ or Pascals (Pa).} \tag{2}$$

There are common alternative units of pressure:

- A *bar* is defined as 100,000 Pa (100 kPa). A pressure of 1 bar is slightly less than the atmospheric pressure on Earth at sea level (approximately 1.013 bar).
- A *pound per square inch* (PSI) is an imperial unit defined as the weight of one pound (454 g) applied to a square inch (2.54 cm)^2 of area.

Pressure is measured using a manometer or a barometer. The pressure in tyres or your veins is found using manometers which compare the pressure of interest to atmospheric pressure, whereas barometers compare the air pressure to a vacuum and so are used to find atmospheric pressure.

Example 2. How many PSIs is atmospheric pressure?

Solution:

$$1 \text{ PSI} = \frac{\text{force from 1 pound}}{\text{area of 1 square inch}} = \frac{0.454 \text{ kg} \cdot 9.81 \text{ N/kg}}{(0.0254 \text{ m})^2} \approx 6900\text{Pa}$$

[4] Normally it is the surface of the wall of the container or the surface of an object that is submerged into that fluid.

Since atmospheric pressure is 101.3 kPa, it is approximately 101,300/6900 = 14.7 PSI. This is a useful number to remember when pumping your tyres.

Pressure is a key concept in the physics of fluids because of a fundamental principle known as *Pascal's law*. This states that *the pressure of a fluid (at a given level) is the same in all directions.* It is attributed to Blaise Pascal — the French scientist who demonstrated many of the effects of hydrostatics in the 17th century.

We experience Pascal's law in everyday life. For example, squeezing a balloon takes the same effort no matter which direction you squeeze on the balloon. When you squeeze a tube of toothpaste at the end, the pressure is transmitted equally through the toothpaste, forcing the toothpaste out through the opening.

Example 3. A closed toothpaste tube with an opening of diameter $d_1 = 0.5$ cm is pressed so a force of $F_1 = 1$ N acts on the cap. Find the force on the cap if you squeeze a different tube with an opening of $d_2 = 0.8$ cm hard enough to produce the same pressure inside.

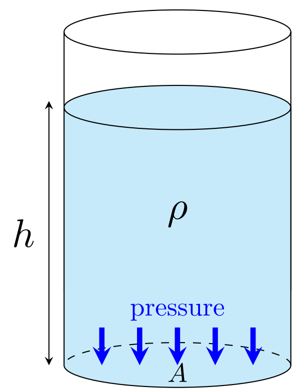
Solution: The pressure on the cap depends on the force you apply, and it is the same in both cases. We find this pressure by dividing the force F_1 by the area $\pi d_1^2/4$ of the first tube opening: $P = \dfrac{F_1}{\pi d_1^2/4}$. The force in the second case is found by multiplying this pressure by the area of the second opening:

$$F_2 = P \cdot \pi d_2^2/4 = F_1 \frac{\pi d_2^2/4}{\pi d_1^2/4} = F_1 \frac{d_2^2}{d_1^2} = 2.56 \text{ N}.$$

The pressure from the atmosphere is barely noticeable to us. It is due to the weight of the atmosphere above us, it also acts equally in all directions. If this were not the case, you would not be able to pick up a piece of paper from a table, as the atmosphere on the top side would be pressing it down with a force that is equivalent to a weight of a kilogram per square centimeter! The air at the sides and under the paper also exert the same pressure allowing you to move it freely about as no net force is pushing on it from the air.

Pascal's Law allows us to think of pressure in a fluid both in terms of how much force is needed to compress it and in terms of how much of a push it can exert on something. The pressure measured after pumping a bicycle tyre is the same whether we determine it from the force required to pump more air in, or from the pressure inside pushing back on a manometer connected to its valve. Hence we can simply say "the pressure inside the tyre is 50 PSI", without thinking of the directions of any forces or areas.

2.2 Pressure of a column of liquid



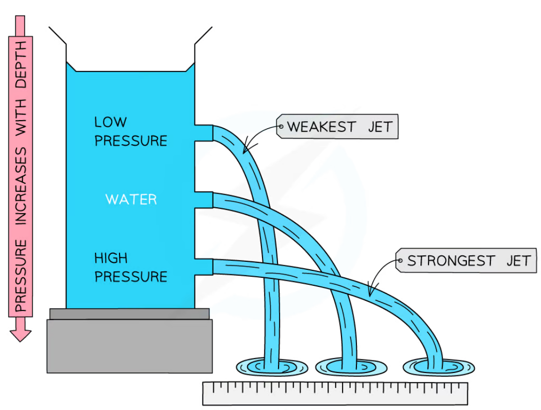
Consider a liquid of density ρ filling a cylindrical vase of height h. Ignoring atmospheric pressure for the time being, what is the pressure on the bottom of the vase due to the weight of the liquid? Recall that weight is the force due to gravity and equals mass $\times$ g, where g $= 9.81\text{ms}^{-2}$. We can use the expression (2) to write

$$P = \frac{mg}{A},$$

where m is the mass of the liquid and A the area of the vase's bottom. Recall from the previous section the expression relating mass, density and volume; $\rho = \frac{m}{V}$, hence $m = \rho V$. We also remember that the cylinder volume, V, is the product of the area of the vase multiplied by the height of the column, $V = A \times h$. Putting all of this together (please do it yourself) we have

$$P = \rho g h. \tag{3}$$

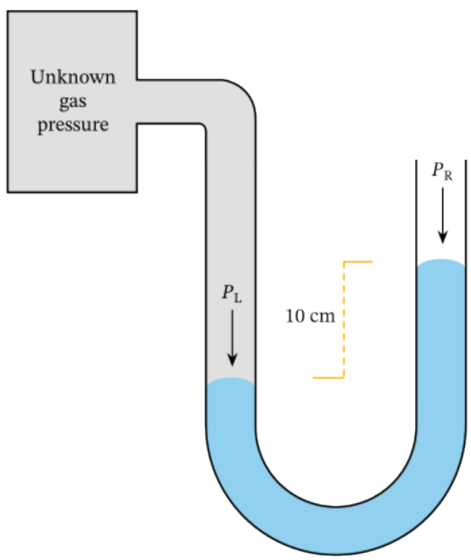
But what if we want to know the pressure on the cylinder's sides? If a cylindrical solid were to fit perfectly inside the vase, the sides of the vase would experience no pressure. But with fluids, the sides are pushed by the molecules of the liquid which according to Pascal's Law exert a pressure equally in all directions. Thus each small part of the side at a particular depth, h, will experience pressure $P = \rho g h$. The diagram below shows how the pressure on a wall increases with depth, pushing the liquid out with a stronger force. An actual experiment to this effect can be found in this Khan Academy video.



The pressure inside a liquid is in addition to atmospheric pressure. However, in most cases it is the *extra* pressure over and above the atmospheric pressure that is of interest. For example, a tyre with a hole is of no use even though it has air inside it at atmospheric pressure. This is because the pressure inside and outside the tyre is the same, so the tyre can be squeezed very easily and doesn't support any extra weight. For this reason, manometers show the excess, rather than absolute, value of the pressure inside a tyre. Another example is found when deep sea diving, because pressure inside our body is equal to atmospheric pressure, it is the added pressure from the seawater that makes us uncomfortable if we dive too deep.

Problem 5 (2 marks). How deep do we need to dive so that the pressure of the water column above us is equal to atmospheric pressure?

Problem 6. (3 marks)



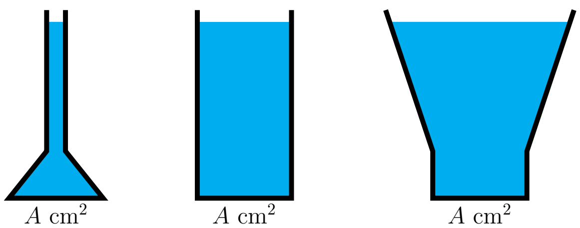
A liquid column manometer contains fresh water, ρ_w=1000 kg/m^3. The left side is connected to a gas of unknown pressure, and the right side is open to the atmosphere at sea level, with pressure of 101.3 kPa. Determine the absolute pressure of the unknown gas.

Problem 7* (4 marks). Find the mass of Venus's atmosphere from the mean atmospheric pressure (1350 PSI) and Venus's radius (6,052 km). Does your result have the same order of magnitude as that of Problem 3? Which answer do you think is more reliable? Why?

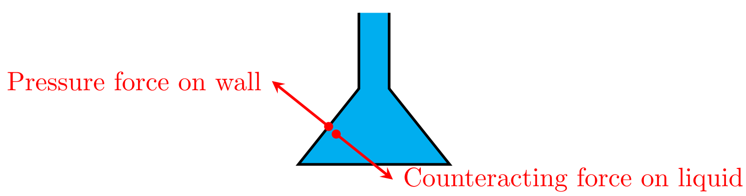
Hint: The surface area of a sphere of radius r is $4\pi r^2$.

2.3 Pascal's paradox

Example 4. An interesting consequence of Pascal's law is the *hydrostatic paradox* or Pascal's paradox. The three containers shown below have the same area A at the bottom and the same height of liquid, so the hydrostatic force $\rho g h A$ at the bottom of the container is the same, even though the volumes, and hence the weights, of the liquid above are quite different. How can this be?



Solution: Consider the sloped walls of the vessel on the left. According to Pascal's Law, the liquid applies outward pressure on these walls. But this means, according to Newton's Third Law[5], that the walls will counteract with a inward force on the liquid. This force is in addition to the liquid's weight and the combined force makes up the total force on the container's bottom.



A similar argument can be made for the third container: the force from the slanted walls *reduces* the force on the bottom. Only for the second container the pressure force on the bottom is equal to the weight of the liquid.

It is important to understand, however, that if the container is put on scales, the scales would be registering the total vertical force of the liquid on *all* walls, not just on the bottom.

Answer:

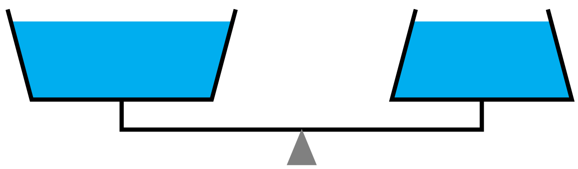
- first container: force on bottom $>$ weight of the liquid;
- second container: force on bottom $=$ weight of the liquid;
- third container: force on bottom $<$ weight of the liquid.

[5] We will cover Newton's Laws in a subsequent assignment, but we assume you have some idea of it from your previous studies.

Pascal's Barrel is an experiment illustrating the hydrostatic paradox. Supposedly, Pascal attached a tall pipe with a water proof seal to a wooden barrel full of water. He filled the pipe and at a certain point, boom! The barrel exploded. A very cool modern version of this is demonstrated here by Dr Katerina Visjnic from Princeton University.

Problem 8 (3 marks). What dimensions are needed if a long, thin circular tube filled with 1 litre of water can shatter a 50l glass flask as in the video clip? Assume that the flask can withstand a pressure up to 7×10^5Pa. State the height of the tube and its radius.

Problem 9 (3 marks, from *Professor Povey's Perplexing Problems*).
A pair of scales has a built in set of vessels as shown in the diagram, one vessel has the sides tapering inwards towards the top and the other tapering outwards. The vessels have the same bottom area and, when they are empty, the scales is balanced. The vessels are filled to the same height with water. According to the laws of hydrostatics, the force on the bottom of each pan should be the same. Thus the scales will remain balanced even though the vessel on the left has more water. Is this logic correct? Explain with the aid of a force diagram for each vessel on the scales.

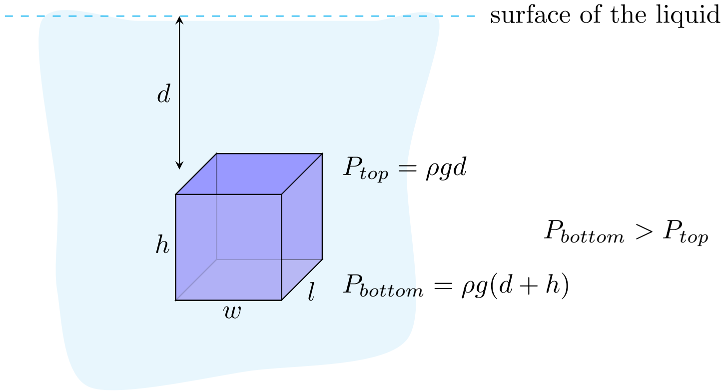


3 Buoyancy and Archimedes principle

3.1 Archimedes' principle

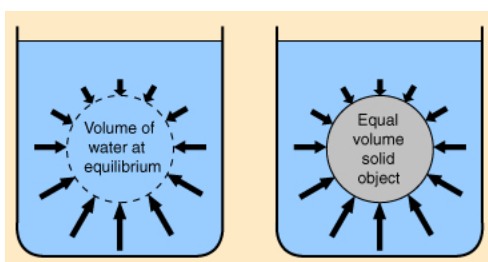
In the previous section we introduced the idea that pressure varies with depth. Any object submerged in a fluid extends across some depth, therefore the object will experience a difference in pressure from the fluid acting on the top of the object compared to the bottom. What do the forces on the object look like and how would this affect the motion of the object?

Example 5. A cuboid of length l, height h and width w is submerged into a liquid of density ρ. Find the resultant pressure force of the liquid on the cuboid.



Solution. Let the top side of the object be at depth d. Then the pressure on that side is $\rho g d$, and the resulting force is $\rho g d l w$, where lw is the area of the top side. The bottom side experiences a greater pressure $\rho g(d+h)$, and the corresponding upward force is $\rho g(d+h)lw$. The pressure forces acting on the sides are pairwise equal and opposite and so there is no resulting force in any sideways direction. Taking the difference between the upward force at the bottom and the downward force at the top, we find the overall upward force $F = \rho g(d+h)lw - \rho g d l w = \rho g h l w = \rho g V$, where V is the cuboid volume.

This net force which acts on any object submerged in a fluid is called *upthrust* or *buoyancy*. It was easy to calculate for a cuboid, but how can we generalize it to an object of *arbitrary* shape? Every element of its surface area experiences a pressure force in a different direction, so summing them up seems to be a daunting task.



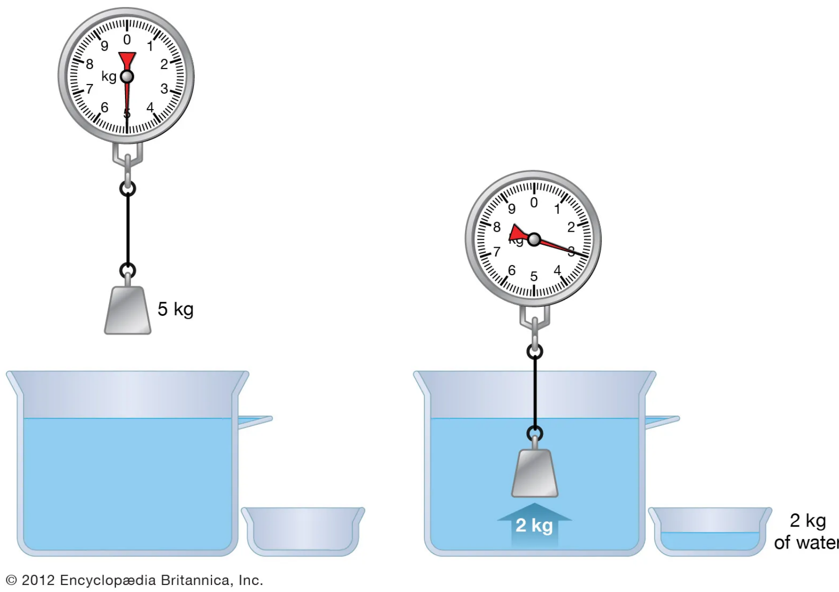
Fortunately there exists a simple and ingenious argument, which is attributed to Archimedes. The diagram shows an object submerged in the liquid (right) and an imaginary volume of liquid of exactly the same shape in place of the object (left). We now note that the forces pushing from the liquid at different depths are the same whether an object or a volume of the liquid is present. Hence the volume of the liquid experiences exactly the same buoyancy force as the object.

In spite of that force, the liquid volume is not moving up or down. This is because the upthrust acting on the liquid volume is exactly balanced by the weight of the liquid. We know that weight is equal to $\rho g V$, where V is the volume of the "submerged" liquid (and of the object) and ρ is the liquid's density. We conclude that the buoyancy force or upthrust is

$$F = \rho g V. \tag{4}$$

This result is often formulated as *Archimedes's principle: The magnitude of the buoyancy force acting on a body is equal to the weight of the fluid that the body displaces.*

Archimedes' principle



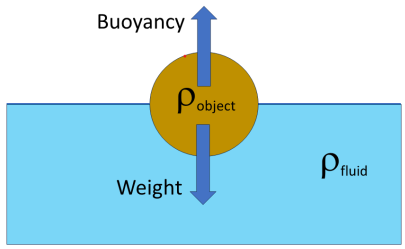
The above image demonstrates that putting the mass into liquid reduces the reading on the scale by 2kg. This means the buoyancy force is approximately equal to 2 kg$\times$9.81 N/kg= 19.6 Newtons. This also means that the weight of the displaced liquid is equal to 19.6 Newtons.

3.2 Flotation

Whether an object floats or sinks depends on its mean density. If this is greater than that of the liquid, its weight will exceed the buoyancy force, so the object will sink. Whether it floats or sinks, upthrust will be acting on the object in the opposite direction to the weight and so the effective weight of an object in a fluid will always be less than its weight out of the fluid.

Problem 10. (2 marks) A 5kg spherical mass of copper with volume 558 cm^3 is suspended from a Newton metre and fully submerged in water from the Dead Sea density 1.24 g/cm^3. By how much does the weight appear to reduce when submerged? Explain why people float easily in the Dead Sea.

If the mean density of the object is less than the density of the fluid, the object will sink just so much as to create enough buoyancy force to push back and balance the weight.



Mathematically, we can write this as follows:

$$\text{weight of object} = \text{upthrust from liquid}$$

$$\rho_{\text{object}} V_{\text{object}} g = \rho_{\text{liquid}} V_{\text{submerged}} g,$$

where $V_{\text{submerged}}$ is the volume of the submerged part of the object. As g is the same on both sides of this expression we can cancel it, then we rearrange the remaining terms for density and volume so they are collected together,

$$\frac{\rho_{\text{object}}}{\rho_{\text{liquid}}} = \frac{V_{\text{submerged}}}{V_{\text{object}}} \tag{5}$$

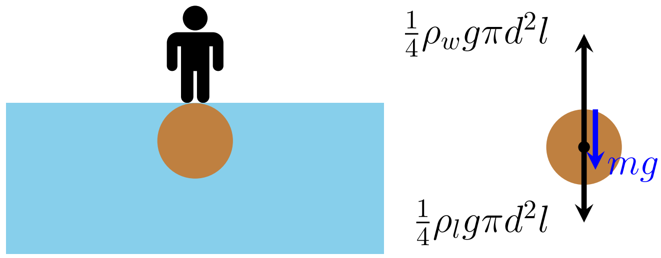
In other words, the density of a submerged object compared to the density of the liquid tells us how much of the object is submerged and how much is still above the surface. We have this relationship:

$$\frac{\text{Density of object}}{\text{Density of liquid}} = \text{fraction of object in the water} \tag{6}$$

Example 6. An iceberg floats in the North Atlantic ocean. The density of ice is 917 kg/m^3 compared to seawater which has a density of 1020 kg/m^3. What volume of water does the iceberg displace if it weighs 3000 tonnes?

Solution: The iceberg has a mass of $m = 3 \times 10^6$ kg, hence its volume is $V_{\text{iceberg}} = m/\rho_{\text{ice}} = 3272$ m^3. To find the submerged volume, we rearrange the relation (5):
$V_{\text{submerged}} = V_{\text{iceberg}} \dfrac{\rho_{\text{ice}}}{\rho_{\text{water}}} = 2944$ m^3, or about 90% of the total volume.

Example 7. A cylindrical log of length $l = 3.5$ m and diameter $d = 30$ cm is floating in fresh water. What is the maximum mass m of a person who can stand on the log without getting their feet wet? The density of the wooden log is $\rho_l = 0.7 \times 10^3$ kg/m^3, the density of the water is ρ_w.



Solution: The volume of the log is $V = \pi d^2 l/4$ and hence its weight is $m_l g = \rho_l V g = \rho_l g \pi d^2 l/4$. This weight, together with the weight mg of the person, are the two downward forces acting on the log. They are balanced by the upward buoyancy force, $F_b = \rho_w V g = \rho_w g \pi d^2 l/4$, where ρ_w is the density of water. Hence we can write

$$\frac{1}{4}\rho_l g \pi d^2 l + mg = \frac{1}{4}\rho_w g \pi d^2 l.$$

We cancel g on both sides and readily find

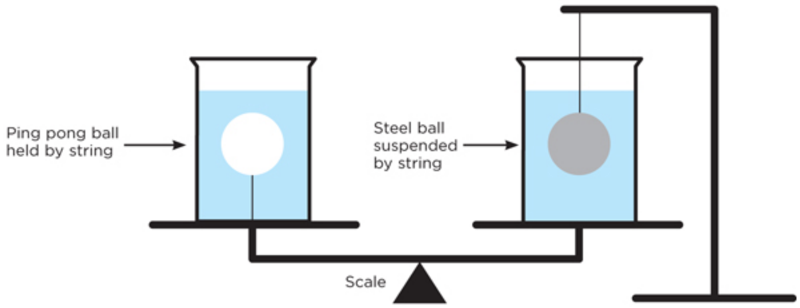
$$m = \frac{1}{4}(\rho_w - \rho_l)\pi d^2 l = 74 \text{ kg}.$$

Problem 11 (4 marks). Archaeologists are recovering a stone pillar of height $h = 2.3$ m and cross-sectional area $A = 200$ cm^2 by winching it up vertically out from under the sea by a rope. The rope tears when a quarter of the pillar's length is above the waterline. Find the maximum tension force T that the rope can withstand. The density of the stone is $\rho_s = 2.6 \times 10^3$ kg/m^3.

Problem 12 (3 marks). A body's weight in water appears to be three times less than it is in a vacuum. What is its density?

Problem 13** (4 marks). An aluminium ball has an empty spherical cavity inside it. When placed into a container with fresh water, the ball sinks and the net force (weight less the buoyancy force) is $F_w = 0.24N$. When placed into petroleum, the ball also sinks and the net force acting upon it is $F_p = 0.33N$. Find the volume of the cavity. The densities of aluminium and petroleum are $\rho_a = 2.7 \times 10^3$ kg/m^3 and $\rho_p = 0.7 \times 10^3$ kg/m^3, respectively.

Problem 14** (4 marks). Two identical containers sit on each pan of scales, each is filled with an equal amount of water. In the left container a ping pong ball is floating attached by a thin thread to the bottom of the container to keep it under the surface, in the right hand container a steel ball of the same size is hung down from a thin wire attached to a stand so it is fully immersed in the liquid. Do the scales stay level, go down on the left or on the right? Explain your answer.



4 Summary

In this assignment we covered

- notions of pressure and density;
- Pascal's law;
- how pressure in a fluid increases with depth;
- buoyancy and Archimedes's principle.

A quick summary of all these ideas can be found here Fluids at Rest: A crash course.