

Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 11

Physics Assignment 01

Ideal Gases

Rachael Hawkins, Alexander Lvovsky

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This is the first Physics assignment for Year 11 from COMPOS. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but don't let this discourage you. Give each problem a go, and skip to the next one if you are stuck. You are encouraged to use the links and any additional reading to help you, and you can come back and attempt any problem as many times as you need.

The questions in each section are arranged in order of increasing complexity, so you should try problems in all sections. Very similar problems will be discussed in our webinars, so think of any questions you would like to ask and share them in the webinar chat. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 34 marks.

1 Gases

Solids, liquids and gases make up the three states of matter. We have looked at liquids in the [Assignment 01 on hydrostatics](#) in Y10 and we learned how solids and liquids react to heat in [Assignment 10 on thermodynamics](#). Now we study the behaviour of gases. We will use concepts of pressure, momentum, energy, and the amount of substance. The last should be familiar to you from your chemistry lessons on moles and molar masses.

1.1 Boyle's, Charles' and Gay-Lussac's laws

A gas is a state of matter where the particles are very far apart, they move at high speed, and interaction between the molecules is weak. Thus a gas will expand to fill any volume of a container of any shape, albeit at increasingly lower density as the volume expands. The behaviour of gases under different conditions has been studied since the 1600s. Notable advancements have been made by Robert Boyle, followed by French chemists Jacques Charles and Joseph Louis Gay-Lussac, who discovered the interdependence of the volume, pressure and temperature of a gas. Here are the modern formulations of these relations.

- **Boyle's Law.** For a fixed amount of gas at a fixed temperature, increasing the volume V causes the pressure P to decrease — these quantities are inversely proportional.

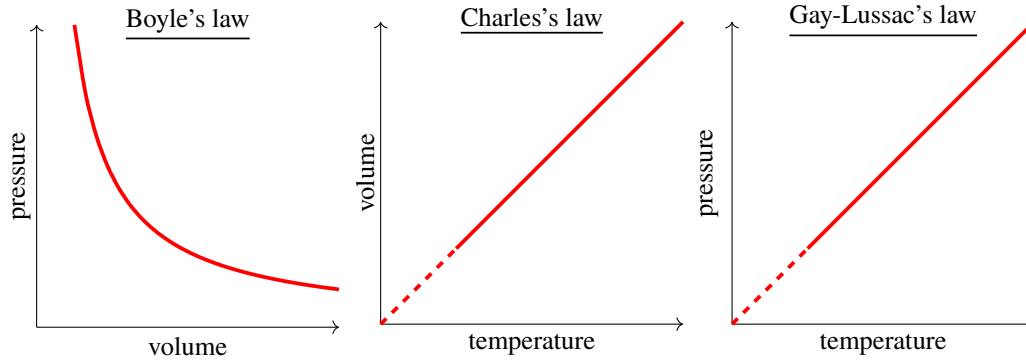
$$P = \frac{\text{constant}}{V} \implies PV = \text{constant}.$$

- **Charles' Law.** For a fixed amount of gas at a fixed pressure, increasing its absolute temperature T will proportionally increase its volume V .

$$V = \text{constant} \times T.$$

- **Gay-Lussac's Law** For a fixed amount of gas at a fixed volume, increasing its absolute temperature T will increase its pressure P .

$$P = \text{constant} \times T.$$



Two of these relationships can be easily demonstrated with a balloon if you tie a good knot such that no gas can enter or escape. You know that if you squeeze on the sides of a balloon — and thus decrease the volume — the balloon exerts a force on your hands due to increasing pressure (Boyle's law: the gas inside is of the same temperature as the air outside). Increase the pressure too much and the rubber will break!

If you take this balloon and stick it under some ice cold water for a minute or so, such that the air inside has become as cold as the icy water, and take it back out, you will see the balloon is much smaller than when you put it in (Gay-Lussac's law). As the air inside the balloon heats back up, the balloon will slowly expand to its original volume.

To see a manifestation of Charles' law, measure the tyre pressure of a car before and after a long drive. Driving (especially the usage of brakes) will heat up the air in the tyres, leading to increased pressure.

Example 1. A bike pump is a vertical cylinder $h = 0.5$ m high and $r = 2$ cm in radius. Sahir lifts the plunger to its top position and then plugs the end so no gas can escape. He then releases the plunger. By what distance Δh will it sink under the action of gravity? The mass of the plunger is $m = 0.5$ kg. Assume constant temperature of the air inside the pump; neglect friction.

Solution: This is an example of Boyle's Law in action. The initial pressure of the gas inside the cylinder is $P_1 = 1$ atm = 10^5 Pa; it is equal to the atmospheric pressure outside. After the plunger is released, it drops a few centimeters, so the volume inside the cylinder reduces and pressure increases, becoming P_2 . The upward force $\pi r^2 P_2$ exerted by this pressure must be equal to the pressure force exerted by the outside atmosphere $\pi r^2 P_1$ plus the plunger weight mg .

$$\text{Hence } P_2 = \frac{\pi r^2 P_1 + mg}{\pi r^2} = 1.04 \cdot 10^5 \text{ Pa.}$$

The initial and final volumes of the cylinder $V_1 = \pi r^2 h$ and the final volume is $V_2 = \pi r^2 (h - \Delta h)$. Writing Boyle's law $P_1 V_1 = P_2 V_2$, we find

$$\frac{h - \Delta h}{h} = \frac{P_1}{P_2}, \text{ so } \Delta h = h \frac{P_2 - P_1}{P_2} = h \frac{mg}{\pi r^2 P_1 + mg} = 2 \text{ cm.}$$

You can check this result experimentally.

Problem 1 (2 marks). A can of hair spray has a maximum pressure of 3 atm when full at 25°C and the metal container can withstand up to 3.3 atm internal pressure before bursting. What's the maximum temperature that a full container of hair spray can be exposed to before breaking?

1.2 Amount of Substance

Amedeo Avogadro postulated the following principle which has become known as **Avogadro's law**: Equal volumes of all gases at the same temperature and pressure have the same number of molecules.

You will recall from chemistry the concept of the amount of substance, measured in moles. One mole (mol) of any substance contains the same number of molecules as 12 g of carbon-12, namely *Avogadro's number* N_A :

$$N_A = 6.02 \times 10^{23} \text{ mol}^{-1}.$$

That is, n moles of any substance contain $N = nN_A$ molecules.

Avogadro's Law can be rephrased as the statement that 1 mole of any gas will have the same volume at standard temperature and pressure. It then follows that, for a gas at a fixed temperature and pressure, increasing the amount n of gas (measured in moles) increases the volume V .

$$V = \text{constant} \times n.$$

Restated in simple words, this means that two identical balloons contain twice as many molecules as each of them individually.

Problem 2 (1 mark). Under normal atmospheric conditions, air (mean molecular mass 28.96 amu)¹ has a density of 1.293 kg/m³. What volume does 1 mole of air occupy?

Problem 3* (4 marks). An adult human breathes in and out 1 litre of air at a time. Estimate how many molecules from Julius Caesar's last breath we inhale every time we breathe, assuming that sufficient time has passed for the final breath to mix evenly throughout the Earth's atmosphere. You need to make some reasonable assumptions about the density and depth of the atmosphere.

2 The Ideal Gas Law

In the previous section we introduced four linear relationships between the quantities V , P , T and n .

We can combine all of these together into a single **ideal gas law**:

$$\boxed{PV = nRT} \quad (1)$$

¹Molecular mass is measured in atomic mass units (amu) and it is equivalent to the mass (in grams) of one mole of substance (i.e. 1 amu/molecule = 1g/mol). It approximately equals to the number of nucleons (protons and neutrons) comprising the molecule. For example, molecular mass of air is ca. 29 amu because air is 79% N₂ (molecular mass 28 amu because each nitrogen atom has 7 protons and 7 neutrons) and 21% O₂ (molecular mass 32 amu because each oxygen atom has 8 protons and 8 neutrons).

where $R = 8.31 \text{ J K}^{-1} \text{ mol}^{-1}$ is called the *molar* or *universal gas constant* and is an amalgam of the proportionality constants from each of the 4 laws. Please convince yourself that Avogadro's, Boyle's, Charles' and Gay-Lussac's laws can be derived from the ideal gas law.

$$\frac{PV}{TN} = k_B$$

Recalling that the amount of substance is related to the number of molecules according to $N = nN_A$, we can modify the Ideal Gas Law to

$$PV = Nk_b T, \quad (2)$$

where $k_b = R/N_A = 1.38 \times 10^{-23} \text{ J/K}$ is called the Boltzmann constant.

Underlying this relationship is a set of core assumptions defining the 'ideal gas', which we look at in more detail in the section on Kinetic Theory of Gases below. Real gases deviate from the ideal behaviour, but the ideal gas law works remarkably well in many cases. Let's look at some examples.

Example 2. At a temperature of 0°C and the standard atmospheric pressure of $1.01 \times 10^5 \text{ Pa}$ how much space does one mole of gas occupy?

Solution. We set $n = 1 \text{ mol}$ in the ideal gas law to find the volume V :

$$PV = RT \quad \Rightarrow \quad V = \frac{RT}{P} = 2.24 \times 10^{-2} \text{ m}^3 = 22.4 \text{ liters.}$$

This result is the same as what you should have obtained from Problem 2 using a slightly different method.

Example 3. A hot air balloon has radius $R = 10 \text{ m}$ and the combined mass of the balloon shell, basket and passenger is $m = 100 \text{ kg}$. What temperature does the air inside the balloon need to have for the balloon to start floating? The air outside the balloon is at room temperature.

Solution. The balloon is acted upon by two forces: gravity and the buoyancy (Archimedes) force. When the balloon is just about to start floating, these two forces are in equilibrium. The gravity force is $(m + m_i)g$, where m_i is the mass of air inside the balloon and $V = \frac{4}{3}\pi R^3 = 4200 \text{ m}^3$ is its volume. As we know from the Y10 Assignment on hydrostatics, the buoyancy force acting on an object of volume V submerged in fluid is $F = m_o g$, where m_o is the mass of the fluid displaced by the object, which is equal to the mass of outside air in volume V .

We can find m_i and m_o from the ideal gas law. The amount of substance of gas in volume V is $n = \frac{PV}{RT}$. Using $m = Mn$, where $M = 0.029 \text{ kg/mol}$ is the molar weight of air, we find $m_i = \frac{MPV}{RT_i}$ and $m_o = \frac{MPV}{RT_o}$, where T_i and T_o are the temperatures inside and outside the balloon.

The equilibrium condition is

$$(m + m_i) g = m_o g.$$

If $T_i > T_o$, we have $m_i < m_o$, so the equilibrium is possible. We rewrite the above equation as

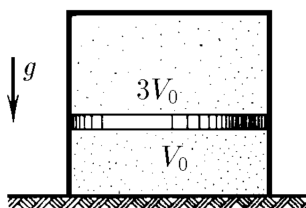
$$m + \frac{MPV}{RT_i} = \frac{MPV}{RT_o}$$

and solve for T_i assuming $T_o = 20^\circ\text{C} = 293\text{ K}$:

$$T_i = T_o \frac{MPV}{MPV - mRT_o} \approx 299\text{ K} \approx 26^\circ\text{C}.$$

Problem 4 (3 marks). An air bubble with a radius of 5.0 mm rises from the bottom of a lake 20 m deep. The temperature at the bottom of the lake is 7°C and the temperature at the surface is 27°C . The atmospheric pressure is 100 kPa. How big will the bubble be when it reaches the surface? The density of water is 1000 kg/m^3 . [The hydrostatic pressure at depth h in a liquid of density ρ is $P = \rho gh$.]

Problem 5 (3 marks). A heavy piston is in equilibrium in a vertical cylindrical vessel with gas as shown in the diagram. The mass of the gas and its temperature above and below the piston are the same. The ratio of the volumes of the upper and lower parts of the cylinder is 3. What will this ratio be if the gas temperature is doubled?



Problem 6* (4 marks). Vessel A containing an ideal gas at 300 K and $5.0 \times 10^5\text{ Pa}$ pressure is connected with a vessel B by a narrow tube, which is initially shut by a tap. Vessel B has four times greater inner volume and contains the same gas heated until its temperature is 400 K and a pressure is $1.0 \times 10^5\text{ Pa}$. How high will the resultant pressure be of the entire system if we open the tube tap while keeping both vessels at their initial temperatures? Neglect friction.

3 Molecular Kinetic Theory

So far we treated the Ideal Gas Law as an experimental fact. However, it can also be derived theoretically by modeling the gas as an (*ensemble*) of molecules flying around with random velocities. This is called the *kinetic theory of gases*.

We begin developing this theory by defining an *ideal gas*. The key property of the ideal gas is that the very large number of identical molecules collide many more times with the walls of the vessel containing the gas than they do with each other. When collisions occur, they are assumed to be *elastic* and don't change the total kinetic energy of the colliding molecules. vt.physics has a short clip which nicely summarises the assumptions that make up the ideal gas model.

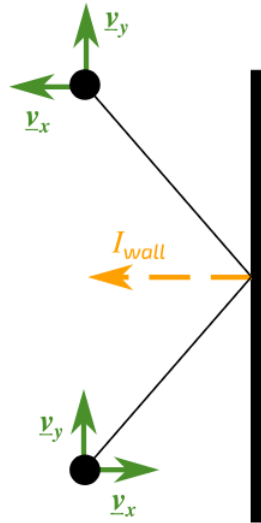
3.1 Kinetic Interpretation of Pressure

It is perhaps intuitive to think that pressure is due to particles repelling each other. Indeed this was the scientific view until the 1600s, when the eminent British scientist Robert Hooke (the discoverer of Hooke's law) realized that pressure is in fact due to the constant bombardment of molecules against the container walls.

Consider a gas confined to a cube with sides of length L^2 . We will focus on the pressure on the wall perpendicular to the x -axis, so molecules move in a positive x direction prior to hitting this wall. When one molecule collides with the wall, the y and z -components of its velocity which are parallel to the wall remain unchanged but the x component will reverse direction³. In the process, the x -component of the molecule's momentum will change according to

$$I_{\text{wall}} = \Delta p_x = p_f - p_i = (-mv_x) - (mv_x) = -2mv_x$$

This momentum change equals the impulse exerted on the wall during the collision.



The time interval for the molecule to travel the length of the cube, hit the wall and return is $\Delta t = \frac{2L}{v_x}$. Over a long period of time t a single molecule will hit the wall $\frac{t}{\Delta t}$ times, exerting a total impulse of $I_1 = |I_{\text{wall}}| \frac{t}{\Delta t} = \frac{m}{L} v_x^2 t$. The total impulse exerted by all N molecules over time t is then the sum⁴

$$I = \sum I_i = \frac{m}{L} \sum_i^N v_{ix}^2 t.$$

These individual impacts occur so frequently that they are perceived as a constant force acting on the wall. As we know, the impulse produced by force F acting during time t is $I = Ft$, so the mean force caused by these impacts equals

$$F = \frac{I}{t} = \frac{m}{L} \sum_i^N v_{ix}^2.$$

Recalling that the wall area is $A = L^2$, we find the pressure of this force:

$$P = \frac{F}{A} = \frac{m}{L^3} \sum_i^N v_{ix}^2 = \frac{m}{V} \sum_i^N v_{ix}^2, \quad (3)$$

because $L = V^{1/3}$ is the cube volume.

²If you prefer listening to reading, the argument below is also presented in this [video by vt.physics](#).

³This section draws on the idea of vector components which was introduced in Y10 Maths Assignment 04. MIT OpenCourseWare [review of vectors](#) has several short videos which very clearly explain what vector components are.

⁴Please watch this [Khan Academy video](#) if you are not familiar with this summation notation.

We don't know the speeds of each individual molecule, v_{ix} , but as we are dealing with a very large number of molecules we can make use of an average molecular speed denoted by a bar above the symbol, \bar{v} . Let's express the above result in terms of the *mean square velocity* which we define as

$$\bar{v}_x^2 = \sum_i v_{ix}^2 / N$$

of all N molecules in the gas. This definition tells us the average velocity in the x -direction of N molecules of this gas. Substituting into Eq. (3) we obtain

$$PV = Nm\bar{v}_x^2. \quad (4)$$

This is starting to look like the ideal gas law in Eq. (2), but to complete the derivation, we need to argue that $m\bar{v}_x^2 = k_B T$. A proof of this equality is beyond the school syllabus, but is a consequence of a fundamental fact from thermodynamics, in thermal equilibrium the total energy of a system is shared equally amongst all the available forms of energy. This is known as the *equipartition theorem*:

The mean energy associated with every degree of freedom in any object in thermal equilibrium is $\frac{1}{2}k_B T$.

The number of ways a molecule in a gas can move are called the *degrees of freedom*. Movement could be translational, rotational or vibrational. The motion of a molecule in each individual direction — x , y and z — is called a degree of freedom.

Hence the mean kinetic energy for molecules moving in each direction is

$$\frac{1}{2}m\bar{v}_x^2 = \frac{1}{2}m\bar{v}_y^2 = \frac{1}{2}m\bar{v}_z^2 = \frac{1}{2}k_B T.$$

Substituting this into Eq. (4), we obtain the ideal gas law from Eq. (2) $PV = Nk_B T$. This equation is significant because a macroscopic variable, the pressure, is expressed in terms of a microscopic variable, the root mean square speed of the molecules. Note that the result does not depend upon the shape of the container, even though we derived this result assuming it was a cube.

Using our knowledge of vector components we know that velocity of the i th molecule can be expressed as a sum of the components in the x , y and z -directions: $v_i^2 = v_{ix}^2 + v_{iy}^2 + v_{iz}^2$, so an average taken over all molecules will give

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2.$$

According to the equipartition theorem, this mean square molecular velocity equals

$$\bar{v}^2 = \frac{3k_B T}{m}. \quad (5)$$

It is three times the mean square molecular velocity in each dimension.

The quantity $v_{\text{rms}} = \sqrt{\bar{v}^2}$ is known as the *root mean square velocity*, or *rms velocity* of the gas molecules.

Problem 7 (1 mark). Show that, for an ideal gas,

$$P = \frac{1}{3}\rho\bar{v}^2, \quad (6)$$

where ρ is the density.

Problem 8 (2 marks). The speed in m/s of 8 molecules are 200, 400, 500, 500, 800, 900, 1100, and 1500. Find the average speed and the root mean square speed⁵.

Problem 9 (2 marks).

- Find the rms velocity of the air i) N₂, ii) O₂ molecules under normal conditions.
- The escape velocity from the surface of the Earth is 11.2km/s. Estimate the temperature at which the following gases escape from the atmosphere: i) N₂, ii) O₂, iii) He. Why do you think helium is a non-renewable resource?

Example 4. A round hole of diameter $d = 1$ mm is punched in a car tyre of volume $V = 10$ liters pumped to a pressure $P_0 = 3$ atmospheres. Estimate how long it will take the air to escape the tyre.

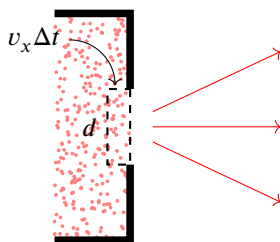
Solution. Consider a short time interval Δt . For the sake of estimation, we can say that the molecules that escape the tyre during that interval are initially localized within a cylindrical volume behind the hole with the height of the cylinder being $v_x \Delta t$, where v_x is the mean thermal velocity in the outward direction. From the equipartition theorem, $mv_x^2/2 = k_B T/2$, so $v_x = \sqrt{k_B T/m} = \sqrt{RT/\mu} \approx 300$ m/s and the volume of the cylinder $\Delta V = \frac{\pi d^2}{4} v_x \Delta t$. Note that μ in the previous expression is the molar mass of the gas⁶.

Amongst all the molecules inside the cylinder, one-half have v_x in the outward direction, and the other half in the inward direction (keep in mind that we are only making a rough estimate). Hence the fraction of the molecules that leave the tyre during time Δt is

$$\frac{\Delta N}{N} \sim \frac{1}{2} \frac{\Delta V}{V} \sim \frac{1}{2V} \frac{\pi d^2}{4} v_x \Delta t.$$

We can estimate the time it takes the tyre to become empty by setting $\Delta N = N$, in which case

$$\Delta t \sim \frac{8V}{\pi d^2 v_x} \sim 80 \text{ s.}$$



Problem 10* (2 marks). Naturally occurring uranium is a mixture of two isotopes ²³⁵U and ²³⁸U. The latter is more abundant with the abundance ratio of 138 (i.e. there is one ²³⁵U atom per 138 ²³⁸U atoms). To produce enriched uranium⁷ with a greater fraction of the desirable 235 isotope, the compound UF₆ uranium hexafluoride is formed and the gas is diffused through a porous material. The rate of diffusion is proportional to the rms speed. What is the isotope abundance ratio in the mixture after the diffusion? The atomic mass of fluorine is 19 amu.

⁵It is easy to prove that the arithmetic average of any set of numbers cannot exceed its rms average. Try to find the proof as an independent exercise. Under which condition are the two averages equal each other?

⁶The molar mass μ is linked to the mass of a molecule m via the equation $\mu = N_A m$. Another useful equation that was used here is $N_A \times k_B = R$

⁷Read more about uranium enrichment methods on the web site of the [U.S. Nuclear Regulatory Commission](#).

Problem 11 (3 marks). A container has two sections separated by a closed valve. Initially, the first section contains a mixture of hydrogen and helium at equal pressures and the second section is evacuated. The valve is opened briefly, so small amounts of the two gases leak into the second section. Find the ratio of pressures of the two gases in the second section after this event.

3.2 Mean Free Path and Brownian Motion

The defining property of an ideal gas is that the molecules don't interact. Of course, molecules of any real gas do collide with each other, but one can say that a gas approaches the ideal model behaviour if the molecular *mean free path* — the mean distance that a molecule travels between collisions — is much larger than the size of the vessel it is contained in. The larger the particles or the denser the gas, the more frequent the collisions are and the shorter the mean free path. If a particle was all by itself, then the mean free path of that particle would be infinite.

This [video by Physical Chemistry](#) derives an estimate for a mean free path of a molecule in a gas:

$$\lambda = \frac{k_B T}{\sqrt{2} \pi d^2 P}, \quad (7)$$

where d is the molecular diameter. One must understand that this relation is only an estimate. A molecule is not a solid sphere but a quantum particle whose wavefunction is smeared in space. Hence even the event of collision is not clearly defined: strictly speaking, molecules have an effect on each other however far apart they are. Equation (7) estimates the mean free path of a molecule between events that *significantly* change the molecule's velocity vectors.

Problem 12 (1 mark). Estimate the mean free path of an air molecule under normal atmospheric conditions, treating the molecule as a sphere of 1 nm diameter.

If you did the calculation correctly, you will find the typical mean free path to be very small, meaning that a gas under normal conditions is far from ideal. Nevertheless, the quantitative results calculated using the ideal gas approximation work surprisingly well, and predict the properties of much denser gases than the model describes.

Problem 13 (2 marks). Consider an *atomic* hydrogen gas under normal atmospheric conditions. How long will it take before 1% of the atoms will pair into molecules H_2 ? Assume that every atomic collision results in emergence of a pair. The atomic radius can be assumed to equal 0.053 nm (Bohr radius).

Problem 14* (4 marks). Estimate the mean free path in a mixture containing two gases. The number densities (numbers of molecules per unit volume) of the gases are d_1 and d_2 , and the molecular radii are R_1 and R_2 respectively.

Hint: The mean free path lengths are different for the molecules of the different gases.