Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 11

Physics Assignment 02

Units, Measurements and Errors

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This is the second Physics assignment for Y11 from COMPOS. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but don't let this discourage you. Give each problem a go, and skip to the next one if you are stuck. You are encouraged to use the links and any additional reading to help you and you can come back and attempt any problem as many times as you need.

The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in our webinars, so think of any questions you would like to ask and share them in the webinar chat. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 29 marks.

1 Units

On 23rd September 1999 the Mars Climate Orbiter satellite, a mission costing a total of $328 million to monitor the Martian atmosphere, went out of control as it unexpectedly passed too close to the planet's surface and either crashed into Mars or shot out into space, it was never heard from again. The cause was a mismatch in units: one piece of software used US imperial units, and another used SI units. As data transferred from one system to the other the calculated impulse from the rocket boosters was out by a factor of 4.45, the conversion factor between the two unit systems.

Understanding what units you are using in a calculation, what accuracy you can give any measurement or calculated quantity and how you represent the uncertainty in those measurements is vital. The credibility of your results depends upon it, not to mention how costly the mistakes can be!

1.1 Système International (SI) Units

The International System of Units is known by the abbreviation SI from its original French name, Système international d'unités. This metric system is the only system of measurement with official status in nearly every country in the world. It is also the standard used in the scientific community.

The SI comprises a coherent system of units of measurement starting with seven *base* units.

SI Base Units			
Base quantity		**Base unit**	
Name	**Typical symbol**	**Name**	**Symbol**
time	t	second	s
length	l, x, r, etc.	meter	m
mass	m	kilogram	kg
electric current	I, i	ampere	A
thermodynamic temperature	T	kelvin	K
amount of substance	n	mole	mol
luminous intensity	I_v	candela	cd

Source: NIST Special Publication 330:2019, Table 2.

All other units used in the SI system can be made from combinations of these base units, and are referred to as *derived* units.

Example 1. Express the following derived units in SI base units

a) Coulomb;

b) Pascal.

Solution.

a) The coulomb (C) is the unit of electrical charge. It is related to two of the base units, time and electrical current via the equation

$$Q = It$$

We combine the units for these quantities in the same way to obtain the unit of charge: C = A × s, the coulomb is defined as one ampere second.

b) The pascal (Pa) is the unit of pressure. It is usually found from the relationship pressure $= \frac{\text{force}}{\text{area}}$. Neither force nor area are base units, they are also derived. To find the correct combination of base units for pressure we must first find what force and area are in base units.

Area is straightforwardly metres squared, m^2. Force is mass times acceleration, so there is a combination of kilograms, metres and seconds equal to one Newton, $F = ma \Rightarrow \mathrm{N} = \mathrm{kgms}^{-2}$.

Combining these to find pressure we have

$$P = \frac{F}{A} \Rightarrow \mathrm{Pa} = \frac{\mathrm{N}}{\mathrm{m}^2} = \frac{\mathrm{kg\,m\,s^{-2}}}{\mathrm{m}^2} = \frac{\mathrm{kg}}{\mathrm{ms}^2} \text{ or } \mathrm{kg\,m^{-1}s^{-2}}$$

Notice how in this example we have moved any quantity with a negative power to the bottom of the fraction where it belongs, and then cancelled like quantities top and bottom following the usual laws of indices.

Problem 1. (3 marks)

Express the following derived units in SI base units

- Watts, W for power
- Volts, V for potential difference
- Ohms, Ω for resistance

Units (dimensions) are very helpful for sanity checks. When you obtain a symbolic answer to a problem, you can find the units of your answer and check if they are consistent with what you expect. For example, suppose your answer for a pressure is $p = mg/x$, where m is a mass, g acceleration and x distance. The SI unit of the right-hand side is $\mathrm{kg}\frac{\mathrm{m}}{\mathrm{s}^2}\frac{1}{\mathrm{m}} = \frac{\mathrm{kg}}{\mathrm{s}^2}$, which is not the correct dimension for the pressure. This should immediately raise a red flag, telling you that you must have made an error.

Always check the dimension of your answer.

Note that standard functions, such as the sine, cosine, exponential, logarithm, etc., only take dimensionless values as arguments. So if you obtain an expression containing e.g. $\sin(t)$, where t is time, a red flag should be raised.

1.2 Conversion of units

Expressing a quantity, which is given in certain units, in different units, which differ only by a constant additive or multiplicative factor is called *conversion*. For example, conversion between temperature in kelvin to temperature in degrees Celsius[1] (see Y10 physics assignment on thermodynamics): $t(\text{in }°\text{C}) = T(\text{in K}) - 273.15$.

Problem 2 (1 mark). When going on holiday with his family for a fortnight, Avi forgot to switch off the lights in his room. Assuming that the light fixture has three bulbs consuming 5 W each, and that electricity costs £0.25 per kilowatt-hour, determine by how much Avi's mistake set back the family budget.

The UK used to use a system of units called *Imperial units*. These are still used with a few minor variations in the USA today. Inches, feet, yards and miles, ounces, pounds and stones, pints, quarts and gallons, and degrees Fahrenheit are examples of Imperial units.

Problem 3 (2 marks). Rephrase the following passage in metric units. Look up conversion factors as needed.

Asha was late for school, a whole 3 miles away. It was a cold morning, only 40°F, so she pulled on a coat and her heavy bag weighing 15 lbs which dug into her shoulder with a pressure of 3 lbs per square inch. She ran out the door at 1.5 yards per second, arriving at school just in time and gasping for air, she quickly drank 20 cubic inches of water.

1.3 Large and small scale units

Physics investigates everything from quarks to the Universe itself. Because it is inconvenient to write long strings of zeros when expressing very large or small numbers, we use powers of ten. We also use prefixes denoting powers of

[1] It is correct to say "degrees Celsius" but "kelvins" rather than "degrees Kelvin".

ten, giving rise to kilometres, nanocoulombs, femtoseconds, etc. The UK Metric Association has a complete summary of prefixes defined by an international convention.

There is a rule for correct usage of the powers of tens notation: the power of ten should be chosen such that the first significant figure[2] immediately precedes the decimal point. For example, the hydrogen atom radius is 0.53Å (where 1Å, or 1 Angström, is 10^{-10} m) or 0.000000000053 meters. Its correct powers of ten expression is 5.3×10^{-11} m. For prefixes, the rules are less strict: we can write this number as 0.053 nanometres or 53 picometres.

Problem 4 (2 marks).

a) A parsec (ps) is distance such that, from an observer 1 ps away, the radius of the Earth's orbit (1 astronomical unit = 149.6 million kilometers) would subtend one angular second (1/3600 of a degree). Express the parsec in meters using both the powers of ten and prefix notations.

b) Express a meter in parsecs using both notations.

c) Find the number of seconds in a year (365.24 days). Hence find the magnitude of 1 light-year in seconds. The speed of light is 299,792 km/s.

d) Proxima Centauri, the star nearest to the Sun, is 4.25 light years away. How many times further away is it from us than the sun is?

2 Errors and Uncertainties

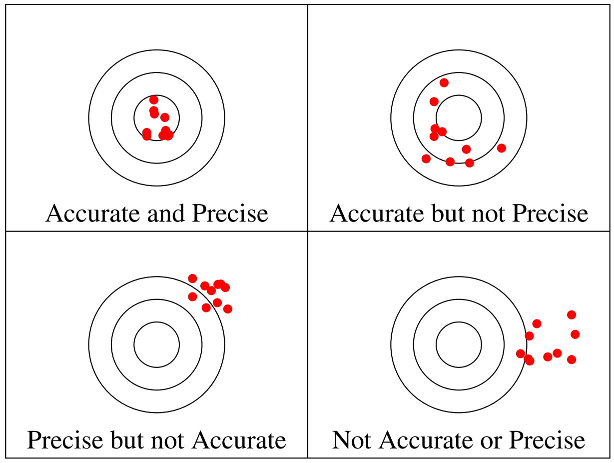
Now we come to the second important topic of this assignment, uncertainties and errors in measurements. When reporting data, errors and uncertainties in measurements are stated alongside actual readings. In experiments, we make efforts to make errors and uncertainties as small as possible.

2.1 Definitions

An *error* in a measurement is the difference between the measured value and the true value. In practice, a physicist may not be aware of the true value of the variable being measured. However, they can estimate the *uncertainty*, or *margin of error*: the range of how far the true value may be from the measured value based on the known properties of the apparatus and observations. In practice, these two terms are used interchangeably.

A related pair of notions is that of *precision* and *accuracy*. Precision is the degree of consistency and agreement among independent measurements of the same quantity, i.e. reproducibility of the result. Precise measurements cluster tightly around a value, but may have an error (*inaccuracy*) as shown in the diagram. This could be due to a badly calibrated piece of equipment, or a human error. We want measurements to be both accurate (distributed around the true value) and precise.

[2] We recall that the significant figures of a number are all its digits except the leading zeros. You can read more on this Third Space page.



Imagine we use an analogue voltmeter with a dial to measure the voltage of a 9V battery. The voltmeter has not been reset to the centre point before taking the reading, and is in fact off-centre by -0.3 volts. Suppose repeated readings of our 9V battery yield values: 8.70 V, 8.72 V and 8.68V. These are precise but inaccurate readings, showing a systematic bias or error.

For the remainder of this section, we will assume that we do not know the true value of the observable being measured. At the same time, we also assume that there is no human error or bias present. In other words, our measurement is accurate but not precise. Our goal is to estimate the measurement uncertainty.

There are two main types of uncertainties. The first type is the accuracy of the measuring apparatus, which is, by definition, a half of its division. For example, a $l = 20$ cm sheet of paper measured by a ruler marked with 1mm divisions has an uncertainty of $\Delta l = 0.05$ cm. We write our measurement result as $l \pm \Delta l = (20.0 \pm 0.05)$ cm.

Note that the number of significant figures we write for l must match the uncertainty. It would be incorrect to write (20 ± 0.05) mm or (20.000 ± 0.05) mm. This convention is useful because it allows one to convey the uncertainty without writing it explicitly. For example, writing $l = 20.0$ cm implies that all these significant figures are known reliably, so the uncertainty must be in the second decimal place[3].

The second type is *random (stochastic) uncertainty*: slight variation in the measurement values due to fluctuations of measurement conditions beyond our control. In the above example with the battery, the three values measured are slightly different due to stochastic uncertainty.

Dependent on the experiment, either the apparatus or stochastic uncertainty may dominate and the other can be neglected. For now, let us focus on the case of the stochastic uncertainty being prevalent. In this case, we must perform multiple measurements and find their *average*. The more measurements we perform, the closer we get to the true value. If the observable being measured is x and we take N measurements with the results $x_1, \ldots, x_N$, then the average is

$$\langle x \rangle = \frac{\sum x_i}{N}.$$

The error of each measurement with respect to the mean is $\Delta x_i = x_i - \langle x \rangle$. How can we estimate the *average* error? Simply adding together all Δx_i and dividing by N will not work: each Δx_i can be either positive or negative, so their mean is exactly zero. One way would be to find the mean of absolute values $|\Delta x_i|$. While this approach is valid, it is not commonly used because modulus is not a smooth function and its mathematical properties are not always elegant.

[3] Admittedly, this rule is not always adhered to in everyday life. Consider, for example, a car with the mass $(2.1 \pm 0.05) \times 10^3$ kg. It is common to write that the mass of such a car is 2100 kg, even though it incorrectly implies that it is known to within ± 0.5 kg, whereas the actual uncertainty is ± 50 kg.

Instead, one finds the *variance* or *mean square error*

$$\left\langle \Delta x^2 \right\rangle = \frac{(\Delta x_i)^2}{N}. \tag{1}$$

The stochastic uncertainty is then given by the square root of this value, also known as the *root mean square (rms) uncertainty* or *standard deviation*:

$$\Delta x_{\mathrm{rms}} = \sqrt{\Delta x^2} = \sqrt{\frac{\sum(\Delta x_i)^2}{N}} \tag{2}$$

The measurement result can then be quoted as $\langle x \rangle \pm \Delta x_{\mathrm{rms}}$.

Problem 5 (2 marks). In an experiment to measure Planck's constant h, the following values were obtained, in units of 10^{-34}Js: 6.65, 6.67, 6.57, 6.61, 6.72, 6.60, 6.66, 6.71, 6.56, 6.54, 6.60. What could you give as:

a) The best estimate of h?

b) The stochastic rms uncertainty?

Problem 6 (2 marks). Suppose we toss a die many times and "measure" the number on its top face. What average and standard deviation do we expect for this measurement?

Hint: If we toss the die N times, each number will occur $N/6$ times on average.

Note a subtlety. If the measurement is dominated by the stochastic error, then by taking an average of multiple samples, one can obtain a very precise approximation of the true value of x, with an error much better than Δx_{rms}. Rather, Δx_{rms} tells the margin of error that a *single* measurement of x has with respect to the true value.

Finally, we need to distinguish *absolute* and *relative* errors. The absolute error is the variable Δx we dealt with above. The relative error is $\delta x = \dfrac{\Delta x}{\langle x \rangle}$. Relative errors are often expressed as a percentage value. In the above example with measuring the length of a paper sheet, we have $\delta l = 0.05/20 = 0.0025 = 0.25\%$.

Problem 7 (1 mark). A snooker ball must be 5.175 cm in diameter to within an uncertainty of ± 0.127 mm. The Earth is 6371 km in radius and its highest mountain above sea level, Mount Everest, is 8848m. Which is smoother in terms of relative error: a snooker ball or the Earth?

Problem 8 (1 mark). Physicists sometimes use the approximation that the speed of light in a vacuum is one foot per nanosecond. What is the percentage error in this approximation? (a meter is 1.094 yard, a yard is precisely three feet).

2.2 Propagation of Errors

In physics it is common for measurements to be combined in some way to produce a quantity of interest. If these measurements are uncertain, what is the uncertainty for the result of the calculation? For example ,the volume of a box is found from multiplying measurements of each side, but each of these have an associated error. What then is the error in the calculated volume of the box?

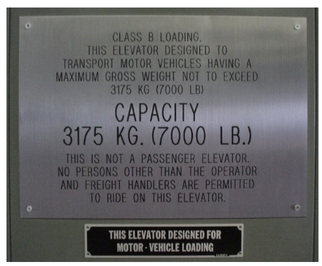
2.2.1 Multiplying by a constant

Suppose we have an experimental quantity x known with the uncertainty Δx. What can be say about the uncertainty of λx, where λ is a number known precisely? Since the true value of x must be within the interval $\langle x\rangle - \Delta x$ and $\langle x\rangle + \Delta x$, the true value of λx must be within the interval $\lambda(\langle x\rangle - \Delta x)$ and $\lambda(\langle x\rangle + \Delta x)$. We conclude that $\Delta(\lambda x) = \lambda(\Delta x)$. We can also say that the *relative* error of λx is the same as that of x.

Problem 9 (1 mark). Jimmy is 180 ± 0.5 cm tall. What is his height in inches and its uncertainty?

Because the relative error is preserved when multiplying by a constant, so is the approximate number of significant figures. So when converting between units, the number of significant figures should remain approximately the same[4]. Please review your solutions to the first part of this assignment to check if you have followed this rule.

Problem 10 (1 mark). Below is a photograph of a plaque in a lift. Apparently, the capacity in pounds was specified by the lift manufacturer, which was converted into kilograms when the plaque was printed. From the physics point of view, what mistakes (if any) were made by the lift manufacturer and the plaque printer?



2.2.2 Addition and subtraction

Suppose we have two measurements: $x \pm \Delta x$ and $y \pm \Delta y$. When we add them, the result will be between $(\langle x\rangle + \langle y\rangle) - (\Delta x + \Delta y)$ and $(\langle x\rangle + \langle y\rangle) + (\Delta x + \Delta y)$, meaning that the sum $x + y$ is known with the uncertainty $\Delta x + \Delta y$.

Example 2. The four sides of a rectangular plot of land are measured using a tape with 1 cm divisions. Find the uncertainty in evaluating the perimeter.

Solution. We sum four values, each of which is known to within ± 0.5 cm. The uncertainty of the result is then $4\times(\pm 0.5$ cm$) = 2$ cm.

For the difference $x - y$, the uncertainty is also $\Delta x + \Delta y$. Please convince yourself of this fact. Avoid a naive error believing that the error is $\Delta x - \Delta y$.

When adding or subtracting two physical quantities, we add their *absolute* uncertainties.

[4]What does "approximately' mean? Suppose we convert a distance of 0.99 meters into yards. This value has two significant figures and implies an absolute error of ~ 0.005 meters, or a relative error of $\sim 0.5\%$. Using the conversion factor of 1.094, we obtain 1.083 yards. If we keep only 2 significant figures, our answer would be 1.1 yards with the implicit absolute error of ~ 0.05 yards, or $\sim 5\%$ — an order of magnitude higher! Therefore it would be more appropriate in this case to keep three significant digits and write that the distance is 1.08 yards. Generally, when in doubt, it is better to keep one too many significant figures than to lose precision.

Before proceeding, let us clarify an important issue associated with the addition of errors. To begin, let us perform a simple experiment.

Problem 11 (3 marks). Toss a pair of dice 20 times and record the values on the top face[5]. Find the mean and rms uncertainty of the measurement for each die as well as the mean and rms uncertainty of the sum of two measurements.

	die 1	die 2	die 1+die 2
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
mean			
variance			
standard deviation			

If you did not make an error in your calclation, you will see that the variance (mean squared error) of the sum is the sum of the variances for each die. One can show that it is always the case.

Problem 12* (4 marks). An experimentalist takes N measurements of observable x with the results $x_1, \dots, x_N$ and N measurements of observable y with the results $y_1, \dots, y_N$. Consider the observable $z = x + y$ and the data set $\{z_i = x_i + y_i\}$. Show that

a) $\langle z \rangle = \langle x \rangle + \langle y \rangle$;

b) $\left\langle \Delta z^2 \right\rangle = \left\langle \Delta x^2 \right\rangle + \left\langle \Delta y^2 \right\rangle$;

c) $\Delta z_{\mathrm{rms}} < \Delta x_{\mathrm{rms}} + \Delta y_{\mathrm{rms}}$ if $\Delta x_{\mathrm{rms}} > 0$ and $\Delta y_{\mathrm{rms}} > 0$ (*Hint:* consider a triangle with the sides $\Delta x_{\mathrm{rms}}, \Delta y_{\mathrm{rms}}, \Delta z_{\mathrm{rms}}$).

We see that the rms uncertainty of the sum is in fact less than the sum of the rms uncertainties of each term. How do we reconcile this with our previous argument that the uncertainties must be added? The intuitive explanation is that, although we have

$$(\langle x \rangle + \langle y \rangle) - (\Delta x + \Delta y) \le x + y \le (\langle x \rangle + \langle y \rangle) + (\Delta x + \Delta y),$$

[5] If you can find only one die, you can throw it 40 times, ascribing the first 20 tosses to die 1 and the next 20 tosses to die 2. You can also use the virtual dice roller.

the “worst case scenario" that both x and y exhibit their highest (or lowest) values simultaneously is quite unlikely. The statistics reflects that, resulting in the mean uncertainty of the sum being slightly lower than the sum of uncertainties.

This being said, measurement uncertainties are typically known only roughly, within an order of magnitude. Therefore, in most applications, uncertainties are simply added. Only in experiments where uncertainties are critical, such as in physical metrology, a more thorough analysis is required.

2.2.3 Division and multiplication

Consider a formula with two or more parts multiplying or dividing each other, for example force $F = ma$ or density $\rho = m/V$. If mass, acceleration and volume have an error associated with each measurement what is the error propagated through to the force or density?

We can derive the relation for multiplication by considering a formula $z = xy$, where $x = \langle x \rangle \pm \Delta x$ and $y = \langle y \rangle \pm \Delta y$ are values with an error. Take the largest possible values for x and y, that is

$$\langle z \rangle + \Delta z = (\langle x \rangle + \Delta x)(\langle y \rangle + \Delta y) = \langle x \rangle \langle y \rangle + \langle x \rangle \Delta y + \langle y \rangle \Delta x + \Delta x \Delta y$$

Usually $\Delta x \ll \langle x \rangle$ and $\Delta y \ll \langle y \rangle$ so that the last term is much smaller than the other terms and can be neglected. Since $\langle z \rangle = \langle x \rangle \langle y \rangle$, the error in z is given by $\Delta z = \langle y \rangle \Delta x + \langle \Delta \rangle y$.

We can write this more compactly by forming the relative error, that is the ratio of $\Delta z / \langle z \rangle$, namely

$$\frac{\Delta z}{\langle z \rangle} = \frac{\Delta x}{\langle x \rangle} + \frac{\Delta y}{\langle y \rangle}.$$

Of course, the percentage error in z can also be written as the sum of percentage errors in x and y.

As discussed above, strictly speaking, we need to add the mean square relative errors of x and y to find the mean square relative error of z. However, we normally simplify our calculations by adding the relative errors linearly.

Example 3. Let $w = (40 \pm 2)$mm be the width of a rectangle and $l = (80 \pm 2)$mm its length. What is the relative and absolute uncertainty in the area of the rectangle?

Solution.

Relative uncertainty for the width $\delta w = \dfrac{\Delta w}{w} = \dfrac{2\text{mm}}{40\text{mm}} = 0.05 = 5\%$

Relative uncertainty for the length $\delta l = \dfrac{\Delta l}{l} = \dfrac{2\text{mm}}{80\text{mm}} = 0.025 = 2.5\%$

Multiplying width by length adds the percentage (relative) uncertainties: $\delta A = \delta w + \delta l = 5\% + 2.5\% = 7.5\%$

The area is of course $A = wl = 40\text{mm} \times 80\text{mm} = 3200\text{mm}^2$.

The absolute uncertainty for the area is $\Delta A = \delta A \times A = 0.075 \times 3200\text{mm}^2 = 240\text{mm}^2 \approx 200\text{mm}^2$ (the absolute uncertainty should be rounded to one significant figure).

Answer: $A = 3200 \pm 200\text{mm}^2$.

Let us now derive the propagation of errors for division. We do it in a similar manner to multiplication: consider a formula $z = \frac{x}{y}$, where $(\langle x \rangle \pm \Delta x)$ and $(\langle y \rangle \pm \Delta y)$ are values with an error. The largest possible value for z occurs when x is largest and y is smallest, that is

$$z + \Delta z = \frac{\langle x \rangle + \Delta x}{\langle y \rangle - \Delta y} = \frac{(\langle x \rangle + \Delta x)(\langle y \rangle + \Delta y)}{(\langle y \rangle - \Delta y)(\langle y \rangle + \Delta y)} = \frac{\langle x \rangle \langle y \rangle + \langle x \rangle \Delta y + \Delta x \langle y \rangle + \Delta x \Delta y}{\langle y \rangle^2 - (\Delta y)^2}$$

Usually $\Delta x \ll \langle x \rangle$ and $\Delta y \ll \langle y \rangle$ so that $(\Delta y)^2$ and $\Delta x \Delta y$ are much smaller than the other terms and can be neglected. The above expression becomes

$$\langle z \rangle + \Delta z \approx \frac{\langle x \rangle \langle y \rangle + \langle x \rangle \Delta y + \Delta x \langle y \rangle}{\langle y \rangle^2}$$

Since $\langle z \rangle = \frac{\langle x \rangle}{\langle y \rangle}$, the error in z is given by $\Delta z = \frac{\langle x \rangle \Delta y}{\langle y \rangle^2} + \frac{\Delta x}{\langle y \rangle} = \frac{\langle x \rangle \Delta y}{\langle y \rangle^2} + \frac{\langle x \rangle \Delta x}{\langle x \rangle \langle y \rangle}$.

The relative uncertainty can be now found:

$$\delta z = \frac{\Delta z}{\langle z \rangle} = \frac{\frac{\langle x \rangle \Delta y}{\langle y \rangle^2} + \frac{\langle x \rangle \Delta x}{\langle x \rangle \langle y \rangle}}{\frac{\langle x \rangle}{\langle y \rangle}} = \frac{\Delta y}{\langle y \rangle} + \frac{\Delta x}{\langle x \rangle} = \delta y + \delta x$$

So, when you have division, the percentage (relative) uncertainties are also added!

When multiplying or dividing two physical quantities, we add their *relative* uncertainties.

For example, if uncertainty in distance d is 3% and the uncertainty in time t is 2%, the uncertainty in speed $v = d/t$ will be $\delta_v = \delta_d + \delta_t = 3\% + 2\% = 5\%$.

Problem 13 (Oxford Physics year 1, 2 marks). The acceleration of gravity can be found from the length l and period T of a pendulum; the formula is $g = 4\pi^2 l/T^2$. Find the relative error in g (i.e. $\Delta g/g$) in the worst case if the relative error in l is 5% and the relative error in T is 2%.

Problem 14 (2 marks). Find the error in the volume of a sphere determined from a measurement of radius $r = 5.3$ cm using a ruler with 1mm divisions. Compare the % error in V to that of r and postulate a rule linking the power to which a value is raised and the final % error.

2.2.4 Errors and derivatives

What shall we do with uncertainties if we have to deal with more complex functions of measured variables? This is where differential calculus can be helpful.

Example 4. As we know, the range of a projectile is $R = \dfrac{u^2 \sin 2\theta}{g}$, where u is the magnitude of the initial velocity and θ is its initial angle above the horizon. Find the uncertainty of R if $u = 10.0$ m/s and $g = 9.81$ m/s^2 are known with a high precision, but the measurement of $\theta = 30°$ has a 2° uncertainty.

Solution. A small change $\Delta\theta$ in the angle will entail a change in the range that equals

$$\Delta R = \frac{\mathrm{d}R}{\mathrm{d}\theta}\Delta\theta.$$

Taking the derivative, we find

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = 2\frac{u^2 \cos 2\theta}{g}.$$

Hence

$$\Delta R = 2\frac{u^2 \cos 2\theta}{g}\Delta\theta \approx 0.3 \text{ m},$$

where we had to convert the uncertainty in the angle into radians: $\Delta\theta = 2° \dfrac{2\pi}{360°} \approx 0.03$ rad.

Problem 15 (2 marks). Solve Problem 14 using derivatives.

When we have a complex function of multiple variables, each of which has an uncertainty, the maths become more complicated: we need to use so-called *partial derivatives*. You will learn about these in Year 13.