Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 12

Physics Assignment 02

Circular Motion

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Due 17th November, 2024

This is the second Physics assignment from COMPOS Year 12. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in tutorials and webinars, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

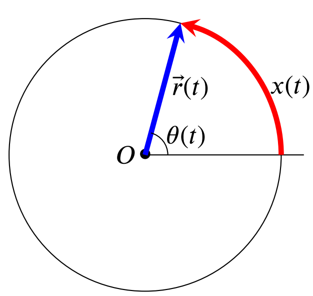
Total 41 marks.

1 Angular motion kinematics

Consider a point object whose motion is restricted to a circular path or radius R around the origin. Let us denote the distance travelled along this path at a given moment in time as $x(t)$. Importantly, this distance is measured along the path, and is not the same as the radius-vector $\vec{r}(t)$ or the displacement $\Delta\vec{r}(t)$. For example, when the object travels one-half of the circle, the distance is $x(t) = \pi R$, not $2R$. In addition, we can introduce the speed $v(t) = x'(t)$ and the *tangential* acceleration $a_\tau(t) = x''(t)$.

Dividing the distance by the radius will give us the angle travelled measured in radians: $\theta(t) = \dfrac{x(t)}{R}$. The first and second derivatives of the angle are called the *angular velocity* [rad/s or s^{-1}] and *angular acceleration* [rad/s^2 or s^{-2}], respectively:

$$\omega(t) = \theta'(t) = \frac{v(t)}{R} \text{ and } \alpha(t) = \theta''(t) = \frac{a_\tau(t)}{R}. \quad (1)$$



The following table illustrates the similarities between the *linear* variables and their *angular* counterparts:

	distance	velocity	acceleration
linear	x	$v = \frac{dx}{dt}$	$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$
angular	$\theta = \frac{x}{r}$	$\omega = \frac{v}{r} = \frac{d\theta}{dt}$	$\alpha = \frac{a_\tau}{r} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Most of this assignment is limited to *uniform* circular motion, such that $v(t) \equiv v$ and $\omega(t) \equiv \omega = \frac{v}{r}$ are constant and $a_\tau(t) = \alpha(t) = 0$. Then $x(t) = vt$ and $\theta(t) = \omega t$.

Here are a few quantities that can be used with uniform circular motion — i.e. movement in a circle with constant speed.

- T — period [seconds], the time for one complete revolution;
- f — frequency [hertz, Hz], the number of revolutions per second.

These quantities are related according to

$$T = \frac{1}{f} \tag{2}$$

Because one full revolution corresponds to an angle of 2π, we have

$$\omega T = 2\pi \tag{3}$$

Combining the last two equations, we find

$$\omega = 2\pi f \tag{4}$$

You can reinforce your knowledge by watching this Khan Academy video.

Although the dimension of both the frequency and angular velocity is inverse time, physicists traditionally use different units for these two variables: Hz and s^{-1}, respectively. This is done to avoid confusion between them. It is incorrect to say, for example, that the frequency of the voltage in the socket is 50 s^{-1} rather than 50 Hz, even though, strictly speaking, a hertz *is* an inverse second!

Problem 1 (1 mark). Calculate the angular velocities, frequencies and periods of

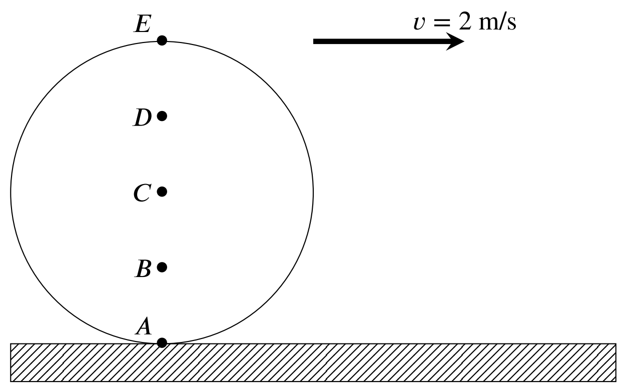
a) the hour hand of the clock;

b) the Earth's rotation on its axis.

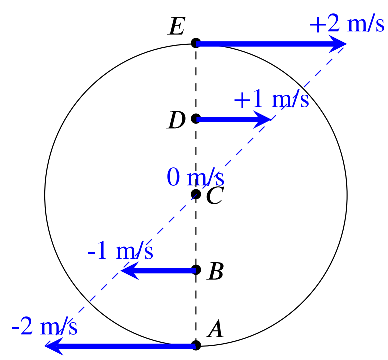
Briefly discuss the difference.

Problem 2 (1 mark). Find the radius of a wheel if the linear speed of a point on the edge is $v_1 = 8$ m/s, and the linear speed of a point $d = 12$ cm away from the edge is $v_2 = 4.8$ m/s. The wheel axis is at rest.

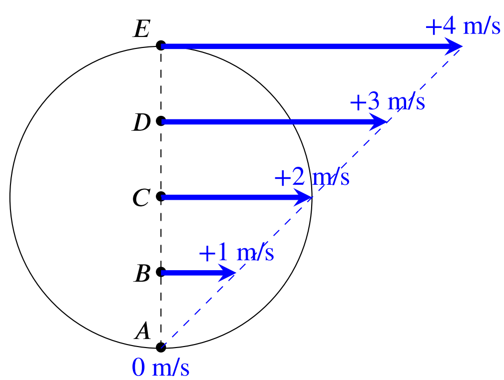
Example 1. A wheel of radius 1 m is rolling without slipping at speed 2 m/s. Find the velocities of points A, B, C, D, E.



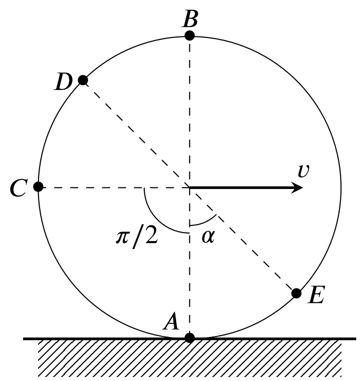
Solution: Let us first work in the reference frame of the car. In this reference frame, the wheel axis is at rest, and all points have the same angular velocity. The linear velocities are then as shown in the diagram.



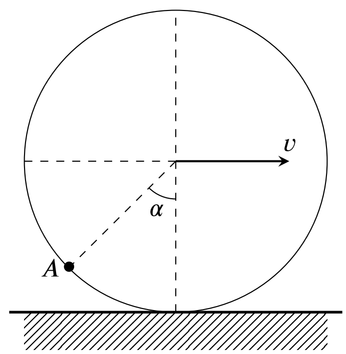
To find the velocities in the reference frame of the earth, we must perform vector addition of the velocities in the reference frame of the car and the velocity of that reference frame with respect to that of the earth (2 m/s to the right). The result is as shown.



Problem 3 (3 marks). A bicycle wheel is rolling along the road without slipping with speed v. Find the velocity (magnitude and direction) of points A, B, C, D, E relative to the ground for an arbitrary angle $\alpha \in [0, \pi/2]$.



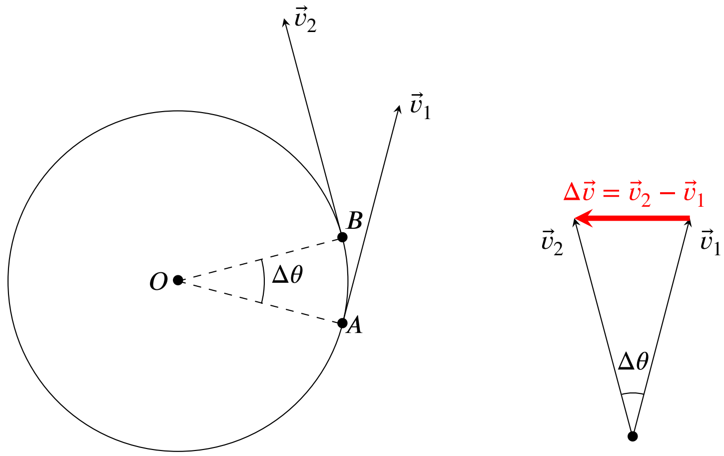
Problem 4 (4 marks). A rock stuck to a wheel of a car moving at speed v flies off at point A as shown in the figure. What maximum height will it reach?



2 Centripetal Acceleration

Even when the circular motion is uniform, the velocity *vector* of each point is not constant, because its direction is changing. The acceleration associated with this change is called *normal* or *centripetal* acceleration $\vec{a}_n$ (or $\vec{a}_c$) and it is directed perpendicular to the motion of the object.

Let us calculate the centripetal acceleration for a point body moving at constant speed v around a circular path of radius r. Suppose it travels from A to B over a short time interval Δt, undergoing a small angular displacement $\Delta\theta$. The diagrams show how the velocity vector changes. From the second diagram, we find the magnitude of the change:



$$\Delta v = |\vec{v}_2 - \vec{v}_1| = 2v \sin\frac{\Delta\theta}{2}.$$

We know that for small angles $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$, so considering a very short time interval where $\Delta t \to 0$,

$$\Delta v = 2v\frac{\Delta\theta}{2} \approx v\Delta\theta.$$

Considering

$$\Delta\theta = \omega\Delta t = \frac{v}{r}\Delta t,$$

we can find the magnitude of the centripetal acceleration:

$$a_c = \frac{\Delta v}{\Delta t} = \frac{v^2}{r}.$$

This can also be written as

$$a_c = \omega^2 r. \tag{5}$$

As we see from the diagram, this acceleration is directed towards the centre of the circle. It is perpendicular to the velocity vector and hence it changes its direction, but not the magnitude.

Problem 5 (3 marks). A car drives at a constant speed v down a spiral driveway (eg. at a multi-level car park) of radius R. The road is inclined at angle α to the horizontal. What is the magnitude and direction of its acceleration?

Hint: Try thinking in the inertial reference frame that moves downwards so the vertical coordinate of the car in this frame does not change.

The centripetal acceleration is orthogonal to the tangential acceleration $\vec{a}_\tau$, which you are already familiar with. The latter equals the rate of change of the *speed*, i.e. is nonzero when the circular motion is not uniform (i.e. when the angular velocity changes). It is directed along the motion of the object, tangential to the circle. The total acceleration is the vector sum of the two:

$$\vec{a} = \vec{a}_\tau + \vec{a}_n.$$

3 Uniform circular motion dynamics

Since there is a centripetal acceleration, according to Newton's Second law, there must be a force that causes this acceleration. This is the *centripetal force*. It is important to understand that this term refers not to a certain physical nature of interaction, but to a type of motion it causes. For example, for a satellite orbiting the earth, the centripetal force is due to gravity, and for a turning car it is due to friction. Please watch this Khan Academy video on how to identify the centripetal force in different scenarios.

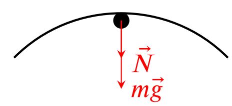
$$F_c = \frac{mv^2}{r} = m\omega^2 r \tag{6}$$

Example 2. A small rock is inside a car tyre of radius $R = 0.4$ m. What is the minimal speed v of the car such that the rock does not move with respect to the wheel? Friction is significant.

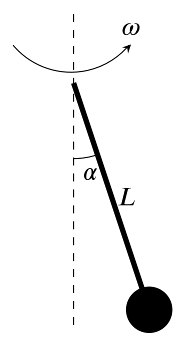
Solution. Suppose the rock is moving with the wheel. Working in the reference frame of the car, consider the moment when the rock is at the top point of the wheel. The forces acting on the rock are its weight mg and the normal reaction force, $\vec{N}$, both downwards. These forces create the centripetal acceleration causing the rock to follow the circular trajectory:

$$mg + N = m\frac{v^2}{r}.$$

The higher the speed, the stronger reaction force is required to satisfy this condition. For minimal v, we have $N = 0$, so $v = \sqrt{gR} \approx 2$ m/s.

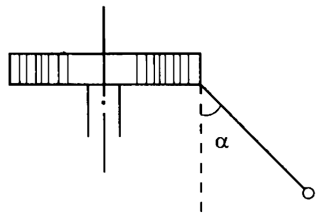


Problem 6 (2 marks).



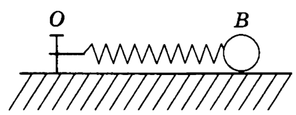
A mass on a light inextensible string of length L is performing circular motion in a horizontal plane. Find the frequency of the motion if the string makes an angle α with the vertical.

Problem 7 (3 marks). A *plumb line* of length $L = 12$ cm is attached to a horizontal disk. When the disk rotates around the vertical axis, the string makes an angle 45° to the vertical. The radius of the disk is $d = 16$ cm. Find the angular velocity of the disk.



Problem 8 (3 marks). A person steps on a set of bathroom scales on the north pole. The reading is $F = 981$ N. What would the scales show for the same person (a) on the Equator and (b) in Oxford? Assume the Earth to be a perfect uniform sphere with the radius $R = 6400$ km. The latitude of Oxford is $\theta = 51°$.

Example 3. One end of a light spring is attached to nail O, and the other is attached to a ball B of mass m. The ball performs circular motion with speed v around the nail with no friction against the table. Find the radius of the circular path. The unstretched length of the spring is l_0. It is known that, when mass B is attached to the spring hung vertically, the length of the spring doubles.



Solution: When the mass is hung vertically the spring extends by l_0, which lets us calculate the spring constant k:

$$mg = kl_0 \Rightarrow k = \frac{mg}{l_0}.$$

When the mass is rotating around the nail the spring provides the centripetal force.

$$\frac{mv^2}{l} = kx, \text{ where } x = l - l_0.$$

$$\frac{mv^2}{l} = \frac{mg}{l_0}(l - l_0).$$

This can be rearranged:

$$gl^2 - gll_0 - v^2 l_0 = 0.$$

This is a quadratic equation with l unknown, whose roots are

$$l = \frac{l_0}{2} \pm \sqrt{\frac{l_0^2}{4} + \frac{v^2 l_0}{g}}$$

The answer with a minus does not make sense as the spring length cannot be negative, so

Answer: $l = \frac{l_0}{2} + \sqrt{\frac{l_0^2}{4} + \frac{v^2 l_0}{g}}$.

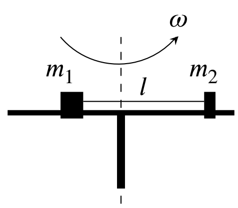
Problem 9 (4 marks). Two small masses m_1 and m_2 can smoothly slide along a horizontal bar. The masses are connected by a light inextensible string of length l. The bar is set into rotation with angular speed ω in a horizontal plane.

a) At what distances from the axis will the masses be in equilibrium?

b) What is the tension in the string?

c) What is the kinetic energy of the masses?

d) Is the equilibrium stable?



You have likely heard the term "centrifugal force", and even more likely experienced it: for example, when riding in a car along a curved path, we feel being pushed in the outward direction. Let us try and understand the nature of thus sensation.

According to Newton's first law, an object not subject to any forces will move with a constant velocity. However, this law applies only in *inertial* reference frames. But what if our reference frame is not inertial? In this case, a free object will accelerate in the direction opposite to the reference frame's acceleration. If the car experiences normal acceleration $\vec{a}_c$ towards the centre, the object's acceleration in the car frame will be $-\vec{a}_c$, i.e. *away* from the centre.

This acceleration is not caused by any force. However, in order to uphold Newton's second law even in a non-inertial frame, it is sometimes convenient to think of this acceleration to be caused by a *fictitious* force of magnitude $-m\vec{a}_\tau$ away from the centre. This is exactly what we call the *centrifugal force*[1] .

Coming back to the example of us sitting in the car, the only force that we truly experience is the centripetal force that the car exerts on us, and this is what makes the passengers move in a circle and accelerate *normally*. We do sense this force, but it appears inconsistent with our observation that we are at rest in the reference frame of the car. The way our mind resolves this contradiction is by telling us that there must be another force — the centrifugal force — counteracting the centripetal force.

Again: the centrifugal force does not exist; it is a sensory illusion and a mathematical abstraction caused by the reference frame being non-inertial. However, this abstraction is convenient in some cases. To see this, you can try re-doing Problems 6 and 7 in the rotating frame. Another example is below.

[1] Please refrain from using the word *centrifugal* in any school examination. The examiners will most likely think that you have confused it with *centripetal* and mark you down.

Problem 10 (2 marks). An air conditioned train from London to Oxford is travelling on a horizontal track at $v = 30$ m/s and turning left around a bend of radius $R = 1000$ m. Inside the train some Physics students are experimenting with DIY accelerometers. One student has tied a light helium balloon to the floor, and the other has tied a plumb line to the ceiling. Find the direction in which the balloon and plumb line will be displaced and the angle which each string makes with the vertical.

Example 4. What is the maximum speed at which a motorcyclist can travel around a bend of radius $R = 90$ m on a horizontal road, if the coefficient of friction between the tyres and the road is $\mu = 0.4$? What is the angle between the motorcycle and the road?

Solution: Firstly, it is important to understand that the only force acting perpendicular to the motion of the rider is friction between the tyres and the road. If friction was not present, the motorcycle would travel in a straight line. So the centripetal force is friction.

The vertical forces must be balanced, so $mg = N$, where N is the normal reaction force. The maximum value of the friction force is $F_{\text{fr}} = \mu N = \mu mg$, so

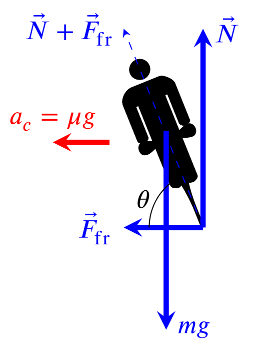
$$\frac{mv^2}{R} \leq \mu mg.$$

$$v \leq \sqrt{\mu g R} \approx 18.8 \text{ m/s}.$$

Note that the maximum speed does not depend on the mass of the motorcycle or rider.

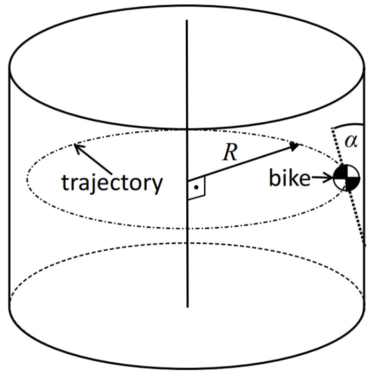
The centripetal acceleration of the rider in the case of the maximum speed is

$$a_c = \frac{v^2}{R} = \frac{\mu g R}{R} = \mu g.$$



To find the angle between the motorcycle and the road, we use the intuition that the resultant of the forces exerted upon the motorcycle by the road — normal and friction must be parallel to — the plane of the wheel (showing rigorously why this is the correct condition is beyond our syllabus). In other words, $\dfrac{N}{F_{\text{fr}}} = \tan\theta \Rightarrow \theta = \arctan(1/\mu) = \arctan 2.5 \approx 68.2°$.

Problem 11 (PAT 2020, 4 marks). As indicated in the figure, a motor bike of total mass m (m is the mass of bike plus rider) is ridden along a horizontal trajectory of radius R on the inside of a cylindrical cage. See video.



The bike and the rider are inclined at an angle α to the wall. The angles α that the bike can make with respect to the walls of the cage are limited by its handle bars to a certain minimum $\alpha_{\min} > 0$. The tyres of the bike have a very high co-efficient of friction with the cage so that the tyres can only roll but not slip along the cage.

(a) Show in a diagram the forces acting on the bike if it is to maintain a horizontal trajectory as shown in the figure.

(b) At what minimum speed v_{min} must the bike travel if it is not to fall down?

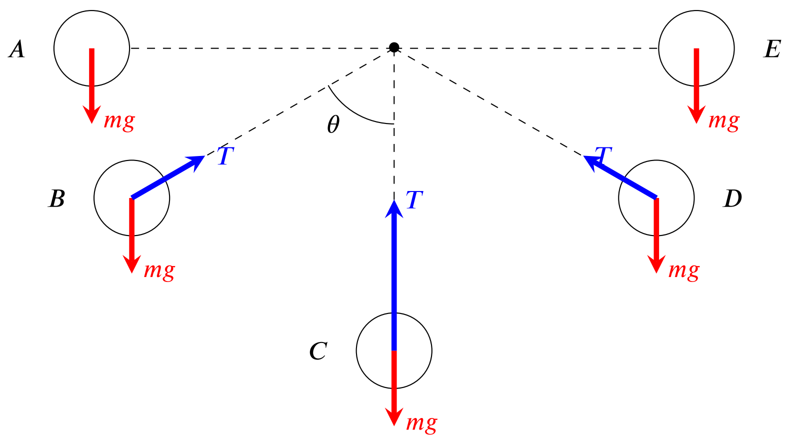
(c) Given that $\alpha_{\min} = 45\ °$, $R = 6$ m, $g = 10$ m/s^2, $m = 200$ kg, find a numerical value for v_{min}.

4 Non-uniform circular motion dynamics

In this section, we consider settings in which both centripetal and tangential acceleration is present. This section of this assignment has the most challenging problems. In most problems you will need to use Newton's 2nd Law and the conservation of energy.

Example 5. One end of a string is tied to a nail, and the other to a small ball. The ball is held so that the string is horizontal and released. At which point(s) of the trajectory will the acceleration of the ball be: a) exactly upwards, b) exactly downwards; c) exactly horizontal?

Solution:



The only forces acting on the ball are gravity mg and tension T of the string. When the ball is first released in position A, the tension is zero and the only force is mg, so the ball will accelerate directly downwards. The similar situation is in position E.

In position C both forces are vertical, so only the centripetal acceleration is present, directed upwards.

Let us now find the angle θ corresponding to the positions B and D where the acceleration is exacltly horizontal. We start by finding T. To this end, we resolve the forces along the string and apply Newton's 2nd Law, keeping in mind that the acceleration along the string is centripetal:

$$\frac{mv^2}{r} = T - mg\cos\theta \tag{7}$$

To find the velocity we will use the conservation of energy:

$$mgh = \frac{mv^2}{2},$$

where $h = r\cos\theta$ is the height difference between points A and B. From the above equation, we have $v^2 = 2gr\cos\theta$. Substituting this into Eq. (7), we find

$$T = 3mg\cos\theta.$$

But on the other hand, in order for the acceleration to be horizontal, we need the vertical components of T and mg be equal and opposite, i.e.

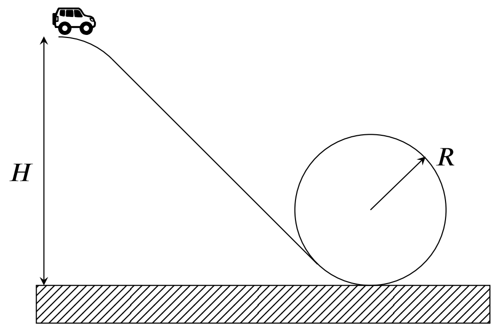
$$T\cos\theta = mg.$$

Comparing the last two equations, we find

$$\cos^2\theta = \frac{1}{3}.$$

Answer: $\theta = \arccos\sqrt{\frac{1}{3}}$.

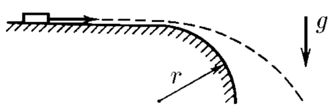
Example 6. A small toy car is rolling without friction along a curved track. What is the minimum height H so that the car passes the loop without losing contact with the track?



Solution: Please watch this video solution by Matt Anderson.

Problem 12 (3 marks). A liana can withstand the tension from a maximum of 2 identical Tarzans hung stationary (see video). Can a single Tarzan swing on this liana up to a 45°angle to the vertical?

Example 7. A block is sliding without friction along a table. What is the minimum speed so that the block, once reached the curved part of the surface, immediately flies off?



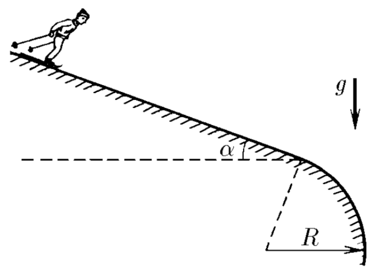
Solution: As soon as the block reaches the curved path, to stay on the path it has to accelerate downwards at $a_c = \frac{v^2}{r}$. Writing Newton's 2nd law for the vertical forces, we have

$$\frac{mv^2}{r} = mg - N,$$

where N is the normal reaction force. If the trajectory is parabolic, it means that the block has lost contact with the surface and it is in free fall. Therefore $N = 0$.

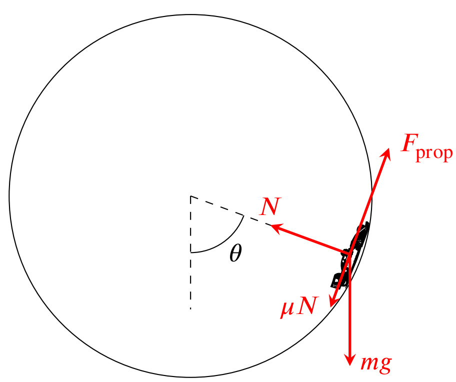
$$\frac{mv^2}{r} = mg - 0 \Rightarrow v = \sqrt{gr}$$

Problem 13 (5 marks). Find the minimum length of the slope (not including the curved bit) needed for the skier to lose contact with the ground immediately after reaching the curved part of the slope. The coefficient of friction between the skier and the snow is μ. Assume $\mu < \tan\alpha$.



Example 8. An *aerosled* (see video) is performing a vertical *loop the loop* at a constant speed v. The coefficient of friction with the snow is μ. Describe how the friction force changes depending on the position of the sled. Sketch a graph of the friction force vs the position of the sled (expressed as the angle θ to the vertical). Find the work done by the force of friction in a single rotation.

Solution: We will start by drawing a free-body diagram with the sled in an arbitrary position θ. The four forces acting on the sled are shown.



Resolving forces normal to the direction of motion:

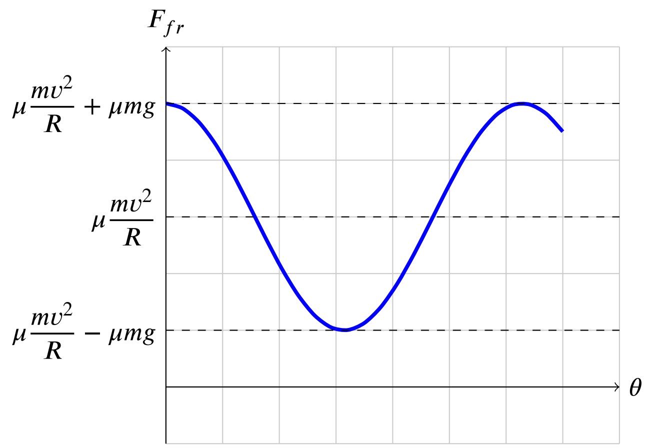
$ma_c = N - mg\cos\theta.$

Resolving the forces along the motion of the sled:

$ma_\tau = F_{\text{prop}} - \mu N - mg\sin\theta.$

Using $a_c = \dfrac{mv^2}{R}$ and $a_\tau = 0$, we find that the force of friction is $F_{\text{fr}} = \mu N = \mu\dfrac{mv^2}{R} + \mu mg\cos\theta$.

It is already clear from this expression that the average friction force is $\mu\dfrac{mv^2}{R}$, and this can be used to find the work done. However, we will go a step further to find the work done by a variable force using integration.



Suppose the sled moves a small distance Δl along the track. The work done will be $\Delta W = F_{fr}\Delta l$, where Δl can be expressed as $R\Delta\theta$, so

$$\Delta W = \left(\mu\frac{mv^2}{R} + \mu mg\cos\theta\right) R\Delta\theta.$$

The work done in a single rotation is the integral

$$W = \int F_{\mathrm{fr}}\,dl = \int_0^{2\pi}\left(\mu\frac{mv^2}{R} + \mu mg\cos\theta\right) Rd\theta = 2\pi\mu mv^2.$$

Problem 14 (3 marks). A small trolley of mass m is rolling without friction along a curved track from the minimum necessary height to complete the loop. What is the contact force F between the trolley and the track at point A? (your answer should be in terms of m, g and θ)