Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 10

Mathematics Assignment 01

The Quadratic Function

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This is the first Mathematics assignment from COMPOS for Y10. The assignment goes into the topic in more detail than you have done in school. There are links to online videos that we encourage you to watch. You are also free to do your own reading around this topic.

This assignment must be your own work. You cannot collaborate with anyone or get help with solving any problem, this includes the use of AI. However, you can use a calculator or any additional resources to research the topic.

This assignment is designed to stretch you, and no student is expected to complete all questions on the first attempt. Some of the problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Harder problems are labelled * or **. You may use calculators for problems that require a numerical solution, but all working out steps must be shown.

Very similar problems will be discussed in the webinars, so think of the questions you would like to ask. We hope that eventually you can solve most of the problems. Good luck!

To successfully complete this assignment, you will need to be familiar with the following formulae:

- Difference of two squares formula:
$$a^2 - b^2 = (a-b)(a+b) \tag{1}$$
- Square of a sum formula:
$$(a+b)^2 = a^2 + 2ab + b^2 \tag{2}$$
- Square of a difference formula:
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{3}$$

An explanation can be found in this Khan Academy web page.

Total 70 marks.

1 Solving quadratic equations

The goal of this assignment is to introduce the concepts of *function* and *quadratic function*. To understand these, we first consider quadratic equations. We will start with intuitive equations that can be solved by inspection.

1.1 Solving by inspection

Example 1. Solve $x^2 = 9$.

Solution: You probably already found a solution: $x = 3$, however, to solve an equation means finding all solutions or showing that there are none. So after a more careful consideration you will find that this equation is also true when $x = -3$. Answer[1] $x = \pm 3$.

The solutions of equations are called *roots*. For example, the roots of the equation $x^2 = 9$ are ± 3.

In the above example, we could have a different expression instead of x.

Example 2. Solve $(2x - 7)^2 = 9$.

Solution: Here we have $(2x - 7)$ instead of x. Let us denote $2x - 7$ as y. Now the equation becomes $y^2 = 9$, so y must be 3 or -3. So now we have $2x - 7 = \pm 3$, which is actually two linear equations: either $2x - 7 = 3$ or $2x - 7 = -3$ which are easy to solve. Answer: $x = 5$ or $x = 2$.

Problem 1 (5 marks). Solve equations:

a) $x^2 = 49$; b) $x^2 = 17$; c) $(x + 4)^2 = 25$;

d) $(x - 7)^2 = 23$; e)* $(3x - 5)^2 = (7x + 19)^2$.

Another intuitive example of a quadratic equation is

Example 3. Solve $x^2 = 5x$.

Solution: We are tempted to divide both sides of the equation by x, which leads to a root $x = 5$. But by doing so, we must be careful not to divide by zero. Indeed, we can see that $x = 0$ is also a valid root, and "lost" it by dividing by x thoughtlessly. Answer: $x_1 = 5; x_2 = 0$.

When dividing both sides of an equation by an expression, always check if this expression can be zero.

As before, in the above example we could have a different expression instead of x.

[1] This answer can also be written as "$x = 3$ or $x = -3$" or "$x_1 = 3$, $x_2 = -3$".

Example 4. Solve $(3x+2)^2 = 5(3x+2)$.

Solution: Let us denote $(3x+2) = y$, so the equation becomes $y^2 = 5y$. From the previous example we know that y must be either 0 or 5. So now we have $3x+2 = 0$ or $3x+2 = 5$. Solving these two linear equations we get the Answer: $x = -\frac{2}{3}$ or $x = 1$.

Problem 2 (8 marks). Solve equations:

a) $x^2 = 2x$; b) $(x+7)^2 = x+7$; c) $(x-8)^2 = 17(x-8)$;

d) $7 - 5x = (5x-7)^2$; e) $7(x-2)^2 = 6x - 12$;

f) $(9-4x)(x+5) = 5(x+5)$; g) $(5-3x)(7x-4) = (7x-4)^2$;

h)* $(11x-2)^2 - (7x+4)^2 = 0$.

Problem 3* (3 marks).

a) Solve the equation $x^2 + y^2 = 0$;

b) Hence, solve the equation $(4x^2 - 25)^2 + (10x - 4x^2)^2 = 0$.

1.2 Solving by factorisation

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0, \tag{4}$$

where a, b, c are real numbers.[2] We shall assume that $a \neq 0$ — otherwise such an equation would not be quadratic. The left-hand side of a quadratic equation — the expression $ax^2 + bx + c$ — is called a *quadratic polynomial*.

Let us first assume that $a = 1$, so the expression takes the form $x^2 + bx + c$. A useful technique that helps *guessing* the solution of a quadratic equation is *factorising* its left-hand side — that is, presenting it as a product of *binomials*

$$x^2 + bx + c = (x+p)(x+q). \tag{5}$$

Example 5. Factorise $x^2 - 13x + 30$.

Solution: As discussed in the video by Mr Chernov, we need to find (guess) two numbers that add to -13 and multiply to 30. The numbers are -3 and -10. Answer: $x^2 - 13x + 30 = (x-3)(x-10)$.

So the quadratic equation

$$x^2 - 13x + 30 = 0$$

[2] This is often written as $a, b, c \in \mathbb{R}$. $\mathbb{R}$ is the set of real numbers, you can watch this Nerdstudy video for more information.

can be rewritten in the form

$$(x-3)(x-10)=0.$$

The left-hand side becomes zero when either $x=3$ or $x=10$, which are the roots of this equation.

More generally, the values of p and q in equation (5) are opposite to the roots of the equation, i.e. $p=-x_1$ and $q=-x_2$. So from now on we will be writing the factorisation in the form

$$x^2+bx+c=(x-x_1)(x-x_2), \tag{6}$$

where x_1 and x_2 are the roots of the polynomial: if x equals either one of these numbers, the expression (6) becomes zero.

Let us make a transformation

$$(x-x_1)(x-x_2)=x^2-(x_1+x_2)x+x_1x_2. \tag{7}$$

Comparing this with Equation (6), we see that

$$x_1x_2=c; \qquad x_1+x_2=-b \tag{8}$$

(the above equalities are called *Vieta's formulae*). That is, we need to guess two numbers that add up to $-b$ and whose product is c.

In the above example, we have $x_1x_2=30=c$ and $x_1+x_2=13=-b$.

Example 6. Factorise $x^2-3x-88$ using Vieta's formulae.

Solution: We need to find two *roots* x_1 and x_2 that add to 3 and multiply to -88:

$$\begin{aligned} x_1\times x_2 &= -88\\ x_1+x_2 &= 3 \end{aligned}$$

The roots are $x_1=-8$ and $x_2=11$.

Using formula (6) we get

Answer: $x^2-3x-88=(x+8)(x-11)$.

Sanity check

Whenever you solve a problem, you should think or simple tests to see if your result makes sense. With equations, this is easy: you substitute the roots you found into the equation and check if the equality holds. In example (5) you have $x_1 = 3$ and $x_2 = 10$, substituting these into the original equation:

$x_1^2 - 13x_1 + 30 = (3)^2 - 13 \times 3 + 30 = 9 - 39 + 30 = 0;$

$x_2^2 - 13x_2 + 30 = 10^2 - 13 \times 10 + 30 = 100 - 130 + 30 = 0.$

You are not required to submit the sanity check as a part of your assignment, but you should *always* do it on your own. Otherwise you may end up submitting a solution with a silly mistake, which could easily be avoided.

Problem 4 (4 marks).

a) Find the roots using Vieta's formulae, clearly showing your working, and hence factorise the expressions:

i) $x^2 - 11x + 21$; ii) $x^2 - 2x - 99$; iii) $x^2 - 1103x - 8888$.

b) Write down the solutions to each equation:

i) $x^2 - 11x + 21 = 0$; ii) $x^2 - 2x - 99 = 0$; iii) $x^2 - 1103x - 8888 = 0$.

Example 7. Write down a quadratic equation with roots -3 and 5.

Solution. Vieta's formulae take the form $c = x_1x_2 = -3 \times 5 = 15$ and $-b = x_1 + x_2 = -3 + 5 = 2$.

Answer: $x^2 - 2x - 15 = 0$

Problem 5 (3 marks). Write down a quadratic equation with roots:

a) 7 and -4; b) $-\sqrt{17}$ and $\sqrt{17}$; c)* $3 - \sqrt{7}$ and $3 + \sqrt{7}$

Importantly, not every quadratic polynomial can be factorised — that is, not every quadratic equation has solutions. For example, the equation $x^2 + 1 = 0$ has no roots. In the next section we will learn how to find out how many roots a given quadratic equation has.

Problem 6 (2 marks). Find b and x_2, if $x^2 + bx - 21 = 0$ and $x_1 = 7$.

Example 8.

a) Factorise $x^4 - 5x^2 + 4$.

b) Solve $x^4 - 5x^2 + 4 = 0$.

Solution.

a) Note that if we substitute $u = x^2$, the expression becomes $u^2 - 5u + 4$, which can be factorised into $u^2 - 5u + 4 = (u-1)(u-4)$. Substituting x^2 back into the expression and applying the difference of two squares formula, we obtain

$x^4 - 5x^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$.

b) $(x-1)(x+1)(x-2)(x+2) = 0$. A product is zero if one of the factors is zero. Hence, we have

$(x-1) = 0$ or $(x+1) = 0$ or $(x-2) = 0$ or $(x+2) = 0$,

which gives the following roots:

$x_1 = 1$, $x_2 = -1$, $x_3 = 2$, $x_4 = -2$.

Problem 7 (4 marks). Solve the equations:

a) (PAT[3] 2009) $x^4 - 41x^2 + 400 = 0$; b) $x^4 + 6x^2 - 91{=}0$; c)* $(x+3)^4 - 23(x+3)^2 + 132 = 0$.

More generally, as can be seen by inspection, the equation of the form

$$(x - x_1) \times (x - x_2) \times \ldots \times (x - x_n) = 0$$

has (only) the roots $x_1, x_2, \ldots, x_n$.

What to do if we have a quadratic equation with $a \neq 1$? the simplest approach is to rewrite the equation in the form

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{9}$$

Example 9. Find the roots of the equation $2x^2 - 3x - 2 = 0$ and hence factorise its left-hand side.

Solution: We divide both sides by 2, obtaining $x^2 - \frac{3}{2}x - 1 = 0$, and use Vieta's formulae to guess the roots:

$$\begin{aligned} x_1 \times x_2 &= -1; \\ x_1 + x_2 &= \frac{3}{2}, \end{aligned}$$

so the roots are $x_1 = 2, x_2 = -\frac{1}{2}$. Hence we have

$x^2 - \frac{3}{2}x - 1 = (x-2)\left(x + \frac{1}{2}\right)$.

However, the polynomial we need to factorise is twice the above polynomial. We can write

[3] Oxford Physics undergraduate admissions test.

$$2x^2 - 3x - 2 = 2(x^2 - \frac{3}{2}x - 1) = 2(x-2)\left(x + \frac{1}{2}\right) = (x-2)(2x+1).$$

Of course, we could also write the answer as $(2x-4)\left(x+\frac{1}{2}\right)$ or leave it as $2(x-2)\left(x+\frac{1}{2}\right)$.

Problem 8 (1 marks). Solve the equation $2x^2 - 9x + 9 = 0$ and factorise its left-hand side.

You can see a neat trick to guess the roots of a quadratic expression with $a \neq 1$ in this Mr Chernov's video.

Problem 9 (2 marks). Generalise Vieta's formulae (8): What are the sum and the product of the two roots of a quadratic equation (4) equal to if $a \neq 1$?

1.3 Solving by completing the square

A complete square is a quadratic that can be expressed in the form $(x \pm a)^2, a \in \mathbb{R}$.

To *complete a square* means to separate a quadratic expression into a complete square and a constant.

For example,

$$\underbrace{x^2 + 4x + 7}_{\text{quadratic}} = x^2 + 4x + 4 + 3 = \underbrace{(x+2)^2}_{\substack{\text{complete} \\ \text{square}}} + \underbrace{3}_{\text{constant}}$$

It is worth memorising the pattern of complete squares (these are obtained using equations (2) and (3)):

- $(x \pm 1)^2 = x^2 \pm 2x + 1$
- $(x \pm 2)^2 = x^2 \pm 4x + 4 \leftarrow$ this formula was used in the example above
- $(x \pm 3)^2 = x^2 \pm 6x + 9$
- $(x \pm 4)^2 = x^2 \pm 8x + 16$
- $(x \pm 5)^2 = x^2 \pm 10x + 25$
- ...

Example 10. Solve $x^2 - 6x + 8 = 0$ by completing the square.

Solution: Comparing the LHS expression with the 3rd equation from the pattern above, we can rewrite the equation as

$$x^2 - 6x + 9 - 1 = 0.$$

Note that it is the same equation, as $9 - 1 = 8$. Hence,

$$(x-3)^2 - 1 = 0;$$

$$(x-3)^2 = 1.$$

Like in Example 2, we can see that $x - 3 = \pm 1$. The equation has two solutions: 4 and 2.

Answer: $x_1 = 4$, $x_2 = 2$.

Generally you can use the formula for completing the square:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c. \tag{10}$$

Please check this equality by applying the square of a sum formula (2) to its right-hand side.

Example 11. Solve $5x^2 + 3x - 2 = 0$ by completing the square.

Solution: We first divide both sides of the equation by 5 to have $a = 1$:

$$x^2 + \frac{3}{5}x - \frac{2}{5} = 0$$

We can now apply formula (10) to the expression in square brackets (noting that $b = 3/5$, $c = -2/5$).

$$x^2 + \frac{3}{5}x - \frac{2}{5} = \left(x + \frac{3}{10}\right)^2 - \frac{9}{100} - \frac{2}{5} = \left(x + \frac{3}{10}\right)^2 - \frac{49}{100}.$$

The equation becomes:

$$\left(x + \frac{3}{10}\right)^2 = \frac{49}{100}.$$

Hence

$$x + \frac{3}{10} = \pm\frac{7}{10}.$$

Answer: $x_1 = \frac{2}{5}$, $x_2 = -1$.

Problem 10 (6 marks). Solve the following quadratic equations by completing the square.

a) $x^2 + 6x + 5 = 0$; b) $x^2 - 18x + 32 = 0$;

c) $12x^2 - 7x + 1 = 0$; d) $8x^2 - 11x + 3 = 0$.

e) $x^2 - 12x = 0$; f) $x^2 - 8x + 17 = 0$.

Problem 11 (2 marks).

Show that formula (10) can be generalized to the case $a \neq 1$ as follows:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c. \tag{11}$$

Problem 12* (4 marks). Solve the following equation: $(x-4)^4 - 3x^2 + 24x - 202 = 0$.

1.4 Solving using the discriminant formula

The solutions to a general quadratic equation are given by the well-known formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{12}$$

You can obtain it from (11). A derivation can also be found in this Khan Academy video.

Read about this method on Isaac Physics and review the example therein.

In Eq. (12), the quantity $D = b^2 - 4ac$ is known as the *discriminant*. It determines the number of roots of the equation:

- if $D > 0$, the quadratic equation has two distinct real roots $x_1 = \frac{-b + \sqrt{D}}{2a}$ and $x_2 = \frac{-b - \sqrt{D}}{2a}$.
- if $D = 0$, the quadratic equation has one distinct real root $x_1 = \frac{-b}{2a}$.
- if $D < 0$, the quadratic equation has no real roots.

Example 12. Find the exact roots of the equation $x^2 + 6x - 3 = 0$.

Solution: Using the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times (-3)}}{2 \times 1} = \frac{-6 \pm \sqrt{36 + 12}}{2} = \frac{-6 \pm \sqrt{48}}{2}.$$

This can be simplified:

$$x = \frac{-6 \pm 4\sqrt{3}}{2} = -3 \pm 2\sqrt{3}.$$

At school, you are normally asked to take an additional step to use a calculator and find the answer to 3 significant figures. In COMPOS maths assignments, we follow the convention normally used in mathematics and leave the answer in this exact (surd) form.

Answer: $x_1 = -3 + 2\sqrt{3}$, $x_2 = -3 - 2\sqrt{3}$.

Regular or decimal fractions?

Suppose you are given the equation $24x = 6$. Should your answer be $x = 1/4$ or $x = 0.25$?

While the two numbers are formally identical, a decimal fraction traditionally implies limited precision — that you don't know the answer beyond the significant figures given. In maths problems, this is usually not the case: the numbers are exact. To indicate this, you should use a regular fraction, writing $x = 1/4$.

In physics, in contrast, numbers are not known precisely. Consider the question: *A tortoise walked 6 meters in 24 seconds. What has been its speed?* Even though this is seemingly the same problem, you should answer "0.25 m/s", to implicitly indicate the precision with which the answer is known. In theoretical equations and symbolic answers, however, you should still use regular fractions. For example, you should write $s = \frac{1}{2}at^2$ rather than $s = 0.5at^2$ — because $\frac{1}{2}$ here is an exact value.

Generally you should avoid mixing regular and decimal fractions or put decimals under a square root. In such cases, you should compute your final answer as a single decimal fraction.

Problem 13 (5 marks). For the following quadratic equations:

1) find the discriminant and determine the number of real roots;

2) find the exact value of the roots.

a) $x^2 + 10x + 2 = 0$; b) $x^2 - 5x - 13 = 0$;

c) $6x^2 - 16x + 11 = 0$; d) $25x^2 + 30x + 9 = 0$.

e) $23x^2 - 32x = 0$;

Problem 14* (4 marks). Find all the values of q such that equation

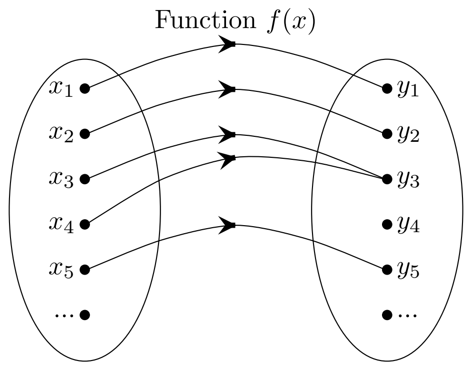
a) $qx^2 - 4x + q = 0$; b) $3qx^2 - (3q + 1)x + 6q - 1 = 0$

has exactly one root.

2 The Quadratic Function

Consider the expression $y = ax^2 + bx + c$. For each x, it calculates the value in the right-hand side and denotes it as y. We can think of this as a *rule*, which for each x evaluates the corresponding value of y. In other words, it *maps* a set of real numbers $\{x\}$ onto a set of real numbers y. In mathematics, such a rule is called *function*. To clarify that an expression should be used as a function, we use notation $f(x)$. For example, writing $f(x) = ax^2 + bx + c$ tells us that the right-hand side is a function of x, whereas a, b and c are the *parameters* whose values are assumed constant.

You can learn more about functions on this Khan Academy webpage.



The function in the form $f(x) = ax^2 + bx + c$ where a, b, c are real numbers and $a \neq 0$ is known as the quadratic function. The graph of $y = f(x)$ is a *parabola*.

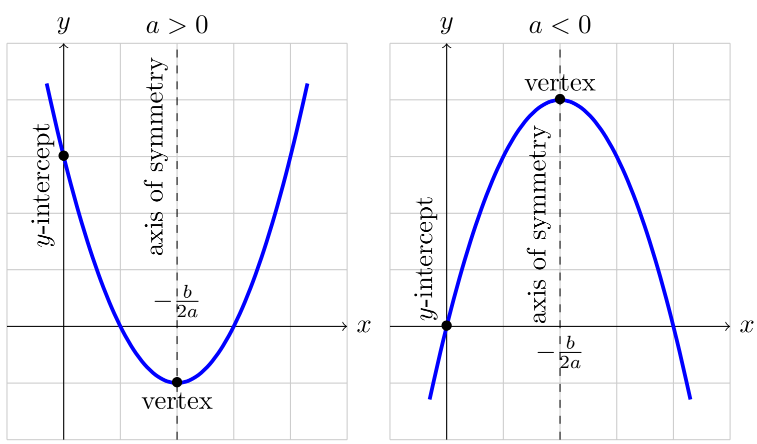


Figure 1: Salient features of a parabola.

The main features of the parabola are as follows.

- The *y-intercept* is the point where the parabola crosses the y-axis, i.e. $(0, f(0)) = (0, a \times 0^2 + b \times 0 + c) = (0, c)$.
- The *x-intercepts* are the points where the parabola crosses the x-axis. Their x-coordinates are the two roots (12) of the equation $ax^2 + bx + c = 0$.
- The *vertex* is the extremal (highest or lowest) point of the parabola. The x-coordinate of the vertex can be found by looking at (11). The squared term in parenthesis is never less then zero, hence the point $x = -\dfrac{b}{2a}$ where it *is* zero is the x-coordinate of the vertex.
- The parabola is *symmetric* about the vertical axis through the vertex. In particular, the x-coordinate of the vertex is located in the middle between the two x-intercepts: $-\dfrac{b}{2a} = \dfrac{x_1 + x_2}{2}$. You can see this from the formula (12).

Please watch the videos on parabolas on this Khan Academy page. Some further information about quadratic functions and parabolas can be found on the AMSI website.

Example 13. Find a,b and c if $M(-1,-7)$ is the vertex and $N(0,-4)$ is the y-intercept of the parabola $y = ax^2 + bx + c$.

Solution: The $x-$coordinate of the vertex is -1, so $-\dfrac{b}{2a} = -1 \Rightarrow b = 2a$. $(0,-4)$ is the $y-$intercept, so $c = -4$.

We also know that when $x = -1$, y is -7, so $-7 = a \times (-1)^2 + 2a \times (-1) - 4 \Rightarrow a = 3$. Hence, $b = 2a = 6$.

Answer: $a = 3$, $b = 6$, $c = -4$, so $y = 3x^2 + 6x - 4$.

Problem 15 (3 marks). Find the parameters a, b, c of the two parabolas in Fig. 1. Assume that the grid step in the figure is 2.

Problem 16 (4 marks). Find the parameters a, b, c of the function $f(x) = ax^2 + bx + c$ if the graph $y = f(x)$ passes through the point $B(8,-9)$ and has a vertex at $A(2,3)$. Hence complete the graph.

Problem 17 (modified PAT 2007, 3 marks). Sketch the curves $y = 2x^2$, $y = 2(x-5)^2$ and $y = 2(x-5)^2 - 9$ on the same set of axes.

Problem 18 (3 marks). A farmer has 200 m of fencing to construct a rectangular pen for his sheep.

a) If he wants to enclose as much grass as possible, what dimensions should he use?

b) The farmer decides to construct the pen next to a long straight stone wall. The wall will act as one side of the pen. What dimensions could he now use to enclose the largest area of grass?

Problem 19 (4 marks). An object is thrown upwards. The height above ground (in metres) can be modelled using the function $h(t) = 30 + 14t - 5t^2$, where the time t is in seconds. Find:

a) the initial height above ground;

b) the maximum height reached by the object;

c) the times at which the object is 34 m above ground;

d) by finding the distance travelled in the first $0.1, 0.01, 0.001$ seconds, work out the initial velocity.