Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 12

Mathematics Assignment 01

Coordinate Geometry and Vectors

Vladlena Kazantseva, Elena Boguslavskaya, David Vaccaro, Vladimir Chernov, Alexander Lvovsky

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This is the first Mathematics assignment from COMPOS Y12. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in webinars (timetable on our website), so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

This assignment should be your own work. You cannot collaborate with anyone or get help with solving any problem, this includes the use of AI. However, you can use a calculator or any additional resources to research the topic.

Total 45 marks.

Our first assignment covers the same material as Y10 assignments 1, 3, 4, 9. It will help you get up to speed if you did not participate in Y10 COMPOS. But even if you did, we have a set of new problems for you that we hope you will find both interesting and challenging!

1 Theoretical Recap

We shall start by compiling a few fundamental results (covered in year 10) on co-ordinate geometry and vectors. You should be already familiar with them, but we provide links to Art of Problem Solving and Khan Academy videos in case you need a reminder. We will go through some of these formulae again in subsequent sections.

1.1 Coordinate Geometry

- The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$
$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$
How to calculate the distance between two points.
- The midpoint (x, y) of a segment with ends $A(x_1, y_1)$ and $B(x_2, y_2)$
$$x = \frac{x_1 + x_2}{2},\ y = \frac{y_1 + y_2}{2}.$$

How to calculate the midpoint of a segment.

- The equation of a straight line
$$y = mx + c,$$
where m is the gradient, and $(0, c)$ is the y-intercept.
How to write the equation of a line in slope-intercept form

- If a straight line makes an angle θ with the positive x-axis (measured anticlockwise from the axis) then the gradient of that line is
$$m = \tan\theta$$

- Equation of a line with gradient m passing through point $A(x_0, y_0)$
$$y - y_0 = m(x - x_0).$$
How to write the equation of a line in point-slope form

- Lines parallel to the x-axis are written as $y = a$, where a is a constant.
Lines parallel to the y-axis are written as $x = b$, where b is a constant.

- Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if and only if $m_1 = m_2$.
Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if and only if $m_1m_2 = -1$.
Equations of parallel and perpendicular lines.

- Equation of a circle with centre at (x_0, y_0) and radius R
$$(x - x_0)^2 + (y - y_0)^2 = R^2.$$
How to write the equation of a circle in standard form.

1.2 Vectors

- The product of a vector with the coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ by a scalar k is a vector
$$k\vec{a} = \begin{pmatrix} ka_x \\ ka_y \end{pmatrix}. \tag{1}$$
When a vector $\vec{a}$ is multiplied by a scalar k, the resulting vector $k\vec{a}$ is *collinear* to $\vec{a}$. If $k < 0$ the vector $k\vec{a}$ is *antiparallel* to vector $\vec{a}$.

  Watch the Khan Academy video: Multiplying a vector by a scalar

- For two vectors with the coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, their sum is the vector given by
$$\vec{a} + \vec{b} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \end{pmatrix}. \tag{2}$$
Please watch the following Khan Academy videos: Adding & Subtracting vectors, Parallelogram rule and Subtracting vectors with parallelogram rule. Review an Isaac Physics example on Describing and adding vectors.

- The scalar (dot) product has two equivalent definitions.

  Definition 1. For two vectors with the known coordinates $\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$, the dot product is the number given by

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y. \tag{3}$$

  Definition 2. For two vectors $\vec{a}$ and $\vec{b}$ with known magnitudes and directions, the dot product is the number given by

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta, \tag{4}$$

  where θ is the angle between vectors.

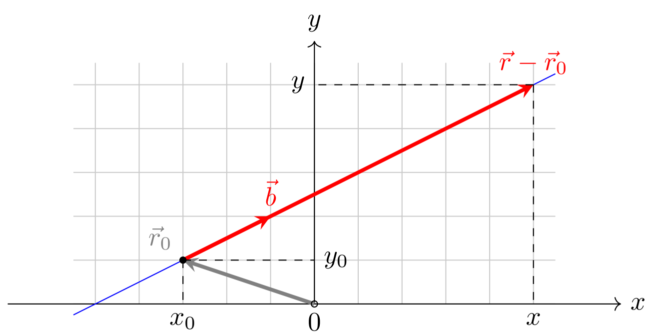
  The equivalence of the two definitions is derived from the law of cosines, as shown in this video by Virtually Passed.

2 Straight lines

2.1 Straight lines and vectors

Vectors are extremely useful in geometry. As an example, please watch this video by MIT OpenCourseWare. It uses vectors to show that the three medians[1] of any triangle intersect at the same point and that this point (known as the triangle's *centre of mass*) divides each median in a ratio of 2:1.

Coordinate geometry and vectors are closely related. We recall that the radius-vector $\vec{r}_A$ of a point A is the vector from the origin to that point. The coordinates $\begin{pmatrix} x_A \\ y_A \end{pmatrix}$ of the radius vector are the same as the coordinates of the point. The coordinates of the vector connecting two points A and B are then $\vec{r}_B - \vec{r}_A = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$.



A straight line passing through a point with the radius-vector $\vec{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ consists of all points $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ such all vectors from $\vec{r}_0$ to $\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ have the same direction. In other words, all these vectors are multiple

[1] A *median* of a triangle is the line connecting one of its vertices to the midpoint of the opposite edge.

of some *direction vector* $\vec{b} = \begin{pmatrix} b_x \\ b_y \end{pmatrix}$. Hence we can write

$$\vec{r} - \vec{r}_0 = \lambda\vec{b} \;\Rightarrow\; \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \lambda \begin{pmatrix} b_x \\ b_y \end{pmatrix},$$

where λ is a number (a scalar). We can rewrite this as a pair of equations

$$\begin{aligned} x - x_0 &= \lambda b_x; \\ y - y_0 &= \lambda b_y. \end{aligned} \tag{5}$$

Dividing both sides of these equations by each other, we obtain the familiar equation of a straight line (in the point-slope form):

$$\frac{y - y_0}{x - x_0} = m,$$

where $m = b_y/b_x$ is the gradient (slope) of the line. We can rewrite the latter equation as

$$\frac{x - x_0}{b_x} = \frac{y - y_0}{b_y}. \tag{6}$$

Parallel lines have direction vectors that are the same (or proportional to each other), meaning that they have the same slope. Perpendicular lines have perpendicular direction vectors. To relate the coordinates of these vectors, let us recall the two definitions of the scalar product of two vectors

$$\vec{b}_1 \cdot \vec{b}_2 = |b_1||b_2|\cos\theta = b_{1x}b_{2x} + b_{1y}b_{2y},$$

where θ is the angle between them. For $\vec{b}_1 \perp \vec{b}_2$, $\theta = \pi/2$ and $\cos\theta = 0$. Hence

$$b_{1x}b_{2x} + b_{1y}b_{2y} = 0 \;\Rightarrow\; \frac{b_{1x}}{b_{1y}} = -\frac{b_{2y}}{b_{2x}} \;\Rightarrow\; m_1 = -\frac{1}{m_2},$$

which is again a familiar result.

An important consequence of the rule for the scalar product is the *cosine law:* In any triangle ABC,

$$BC^2 = AB^2 + AC^2 - 2AB \times AC \times \cos\angle BAC. \tag{7}$$

To prove this, we notice that $\vec{BC} = \vec{BA} + \vec{AC}$, i.e. $\vec{BC} = \vec{AC} - \vec{AB}$, and take the scalar product of each side of this equation with itself:

$$BC^2 = AB^2 + AC^2 - 2\vec{AB} \cdot \vec{AC} = AB^2 + AC^2 - 2AB \times AC \times \cos\angle BAC.$$

Example 1. **(PAT 2010)** Find the equation of the line passing through the points $A(2,3)$ and $B(1,5)$ in the xy plane.

Solution. To find the equation of the line passing through points $A(2,3)$ and $B(1,5)$ calculate the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{1 - 2} = -2.$$

Now we can write the equation using the gradient and the coordinates of one of the points, for example $A(2,3)$:

$$y - 3 = -2(x - 2).$$

Rearranging the terms we obtain $y = -2x + 7$.

Example 2. Find the equation of the line that is parallel to the line $y = 3x + 2$ and passes through the point $r_0 = (5, 5)$.

Solution. The direction vector of the line $y = 3x + 2$ is any vector $\vec{b}$ such that $b_y/b_x = m = 3$, so let's set it to $\vec{b} = \begin{pmatrix}1\\3\end{pmatrix}$. The required line has the same direction vector, so we write its equation using (6) as

$$\frac{x-5}{1} = \frac{y-5}{3} \Rightarrow y = 3x - 10.$$

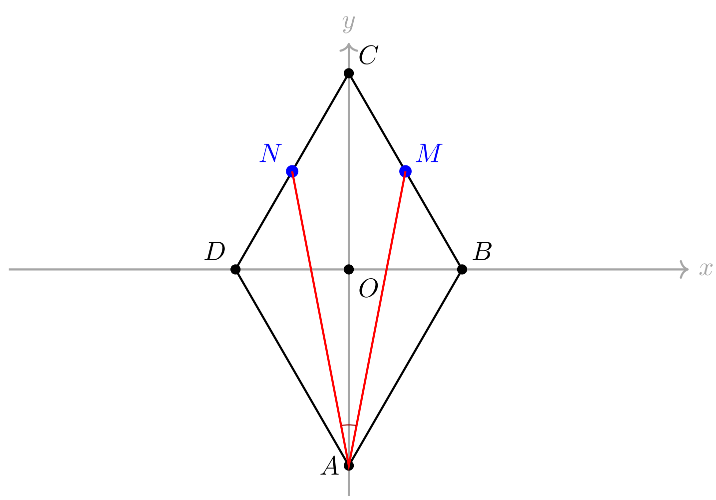
Problem 1 (3 marks).
a) Find the equation of the line that is perpendicular to the line $y = 3x + 2$ and passes through the point $r_0 = (5, 5)$.

b) Find the (shortest) distance between the point $r_0 = (5, 5)$ and the line $y = 3x + 2$.

Problem 2 (3 marks). Points $A(0, 0)$, $B(8, 8)$ and $C(0, 12)$ are vertices of a triangle. Write the equations of the three medians of that triangle, check that they intersect at the same point and find the coordinate of that point. Check that this point divides each median in a ratio of 2:1.

Problem 3 (2 marks). Points $A(2, -1)$, $B(3, 2)$ and $C(-1, 10)$ are corners of an isosceles trapezium $ABCD$. Find the coordinates of D if AB is parallel to DC.

Example 3. $ABCD$ is a rhombus. M and N are the midpoints of BC and CD respectively. Find $\angle MAN$, if $\angle ADC = 120°$.



Solution. Let us set the coordinate axes in the center of the rhombus, as shown in the diagram, and denote the side of the rhombus as a. The rhombus vertices then have the following coordinates:

$A(0, -a\cos 30°)$; $B(a\sin 30°, 0)$; $C(0, a\cos 30°)$; $D(-a\sin 30°, 0)$

or $A\left(0, -\frac{\sqrt{3}}{2}a\right)$; $B\left(\frac{1}{2}a, 0\right)$; $C\left(0, \frac{\sqrt{3}}{2}a\right)$; $D\left(-\frac{1}{2}a, 0\right)$.

The midpoints are hence $M\left(\frac{1}{4}a, \frac{\sqrt{3}}{4}a\right); N\left(-\frac{1}{4}a, \frac{\sqrt{3}}{4}a\right)$

and therefore $\vec{AM} = \begin{pmatrix} \frac{1}{4}a \\ \frac{3\sqrt{3}}{4}a \end{pmatrix}$; $\vec{AN} = \begin{pmatrix} -\frac{1}{4}a \\ \frac{3\sqrt{3}}{4}a \end{pmatrix}$.

We can now use the cosine law or directly the property of the scalar product:

$$\cos\angle MAN = \frac{\vec{AM}\cdot\vec{AN}}{|\vec{AM}|\times|\vec{AN}|} = \frac{13}{14}.$$

It suffices to leave the answer in this form, or we can use a calculator to find $\angle MAN \approx 22°$.

This example shows a neat property of the coordinate method: it can be used to solve problems that are not phrased in the coordinate language. Indeed, almost every geometry problem can be solved using this method (albeit not always elegantly).

Problem 4 (4 marks). $ABCD$ is a parallelogram. $AB = 6\sqrt{2}$, $AD = 16$, $\angle BAD = 45°$, M is a point on CD, $CM : MD = 1 : 2$. N is a point on AD, $AN : ND = 3 : 1$. Find

a) the length of AM;

b) the length of BN;

c) the acute angle between AM and BN.

Problem 5 (2 marks). ABC is an equilateral triangle with side 14. M is a point on BC such that the ratio of the areas of ΔABM and $\Delta ACM = 2 : 7$. Find

a) length of AM;

b) the angle $\angle AMB$.

Problem 6 (3 marks). Find the length of the angle bisector AM of the triangle ABC, if $AB = c$, $AC = b$ and $\angle A = \alpha$.

3 Circles

3.1 Introductory problems

Example 4 (PAT 2013). Find the equation of the straight line that passes through the centres of the two circles:

$$x^2 + 4x + y^2 - 2y = -1$$

and

$$x^2 - 4x + y^2 - 6y = 3.$$

Solution. To find the centres of the circles we need to complete the square and write the equations for the circles in the standard form:

$$x^2 + 4x + y^2 - 2y = -1 \;\Rightarrow\; x^2 + 4x + 4 + y^2 - 2y + 1 - 5 = -1 \;\Rightarrow\; (x+2)^2 + (y-1)^2 = 4,$$

therefore the first circle has the centre at $(-2, 1)$, and for the second, similarly,

$$(x-2)^2 - 4 + (y-3)^2 - 9 = 3,$$

meaning the second circle has the centre at $(2, 3)$. All we need to do now is to write a line that passes through the points $(-2, 1)$ and $(2, 3)$, which is a standard task:

$$y - 1 = \frac{3-1}{2-(-2)}(x - (-2)),$$

which can be rearranged as

$$y = \frac{x}{2} + 2.$$

Problem 7 (1 mark). Find the equation of the straight line which is parallel to the x axis and crosses the circle $x^2 + y^2 = 16$ in points A and B in such a way that $|AB| = 3$. Consider all possible solutions.

Problem 8 (modified PAT 2008, 2 marks). The points $(-8, 1)$ and $(-2, 5)$ are at opposite ends of the diameter of a circle. Determine the equation of the circle.

Problem 9 (3 marks). Let A and B be two points on the plane such that $AB = a$. Show that the set (*locus*) of points M such that $AM = 2BM$ is a circle. Find the radius of that circle and the distance from its centre to points A and B.

Problem 10 (PAT 2018, 2 marks). Determine the area inside the circle defined by:

$$x^2 + y^2 - 8x + 4y + 4 = 0$$

but outside the triangle bounded by the three lines below:

$$\begin{aligned} y &= x - 7 \\ y &= \frac{1}{5}(2x - 29) \\ x &= 7 \end{aligned}$$

3.2 Tangent to a circle

For a point P on a circle, the *tangent* line to a circle at P is a line that intersects ("touches") the circle at P and has no other common points with the circle. The tangent line is always perpendicular to the radius drawn to P. Please study Section 3.2 of Y10 Assignment 3 and examples therein to learn more about tangents.

Problem 11 (2 marks). Find the radius of a circle centred at $(-3, 5)$ if the line with equation $y = 2x + 1$ is tangent to it.

Problem 12 (Skanavi, 3 marks). Three congruent circles of radius r are tangent to each other. For each pair of these circles, a line is drawn that is tangent to both of them but does not cross the third one. Find the area of the triangle formed by these lines.

Problem 13 (MIPT 1989, 4 marks). A circle, whose center lies inside the square $PQRS$, passes through points Q and R. Find the angle between the tangents to the circle drawn from point S, if the ratio of the side length of the square to the radius of the circle is $24/13$.

3.3 Inscribed and circumscribed circles

The circle *inscribed* into a triangle is the circle which is tangent to all three sides of a triangle. The circle *circumscribed* around a triangle passes through all three vertices of the triangle.

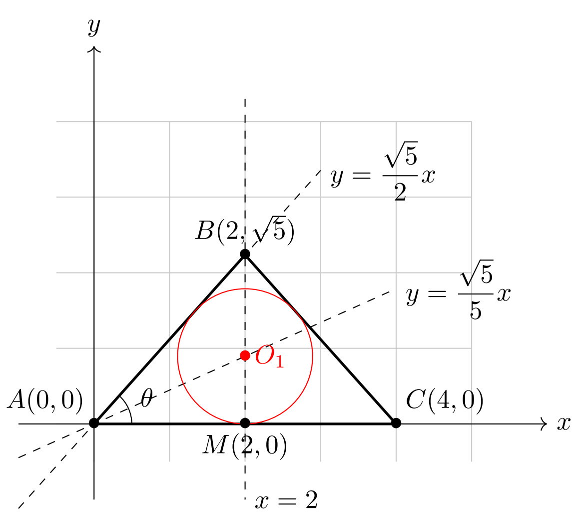
For each triangle, there exists exactly one inscribed and exactly one circumscribed circle. The centre of the inscribed circle is at the intersection of *angle bisectors* of the triangle. The centre of a circumscribed circle is at the point of intersection of the *perpendicular bisectors* of each side. Please make a drawing and convince yourself of these facts. This Krista King Math page may be helpful.

Example 5. For an isosceles triangle with sides 4, 3, 3 find the:

a) radius of the inscribed circle;

b) radius of the circumscribed circle.

Solution. We start by choosing a coordinate system and drawing a diagram.

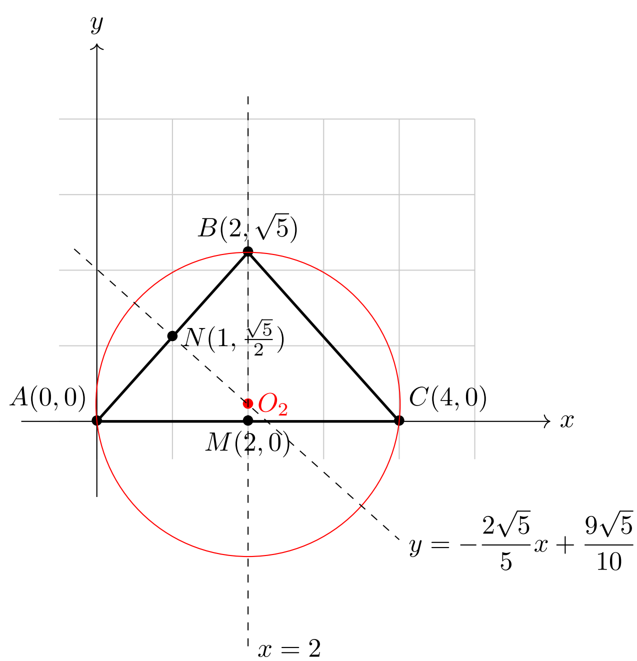


As we know, the centre of the inscribed circle is at the intersection of angle bisectors. Let us find the equations of two of the bisectors. One of them is easy: the bisector BO_1 of $\angle B$ is vertical and has equation $x = 2$.

Let us now find the bisector AO_1 of $\angle A$. We find from the Pythagoras' theorem that the height of the triangle is $\sqrt{3^2 - 2^2} = \sqrt{5}$, which gives us the y coordinate of point B. Hence $\tan\theta = \dfrac{\sqrt{5}}{2}$. Using the trigonometric formula for the tangent of the half-angle: $\tan(\theta/2) = \dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\sqrt{5}/3}{1+2/3} = \dfrac{\sqrt{5}}{5}$. This is the slope of AO_1, so its equation is $y = \dfrac{\sqrt{5}}{5}x$.

Solving it simultaneously with $x = 2$ we find $O_1(2, \dfrac{2\sqrt{5}}{5})$. The radius of the inscribed circle is equal to the length of O_1M, i.e. $r_{\text{insc}} = \dfrac{2\sqrt{5}}{5}$.

To find the radius of the circumscribed circle we will use the same coordinate system. Let us denote the centre of the circle as O_2:



We use the same approach, but now we need the equations of two perpendicular bisectors. One of them — the perpendicular bisector of AC — is $x = 2$

The perpendicular bisector to AB has the gradient $-\dfrac{2}{\sqrt{5}} = -\dfrac{2\sqrt{5}}{5}$ because it is perpendicular to AB. $N\left(1, \dfrac{\sqrt{5}}{2}\right)$ is the midpoint of AB. Using the point-slope formula we find the equation of line NO_2:

$$y - \frac{\sqrt{5}}{2} = -\frac{2\sqrt{5}}{5}(x-1) \;\Rightarrow\; y = -\frac{2\sqrt{5}}{5}x + \frac{9\sqrt{5}}{10}.$$

To find the point O_2, substitute $x = 2$: $O_2\left(2, \dfrac{\sqrt{5}}{10}\right)$

The radius of the circumscribed circle is equal to the length of BO_2: $R_{\text{circ}} = \sqrt{5} - \dfrac{\sqrt{5}}{10} = \dfrac{9\sqrt{5}}{10}$.

Answer: $\dfrac{2\sqrt{5}}{5}$; $\dfrac{9\sqrt{5}}{10}$.

Problem 14 (2 marks). Write down the equation of the circle that passes through the vertices of the triangle created by line $3x + 2y = 12$, x-axis and y-axis.

Problem 15 (3 marks). Find the equation of the circle inscribed in the triangle with vertices $A(-1, 0)$, $B(7, 0)$ and $C(3, 2\sqrt{5})$.

Hint: You may notice a special property of the triangle that simplifies the calculation.

Problem 16 (Skanavi, 2 marks). The centre of the circle circumscribed about an isosceles trapezium with bases 20 and 12 cm is located on its larger base. Find the lengths of the side and diagonal of the trapezium.

Problem 17 (Moscow State University 1989, 4 marks). In trapezium $KLMN$, sides KN and LM are parallel, with $KN = 3$ and $\angle M = 120°$. The lines LM and MN are tangent to the circle circumscribed about triangle KLN. Find the area of triangle KLN.