Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 13

Mathematics Assignment 01

Differentiation 2

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This is the first Y13 Mathematics assignment from COMPOS. This assignment is designed to stretch you and no student is expected to complete all questions on the first attempt. The problems are hard, but do not let this discourage you. Give each problem a go, and skip to the next one if you are stuck. The questions in each section are arranged in the order of increasing complexity, so you should try all sections. Very similar problems will be discussed in tutorials and webinars, so think of the questions you would like to ask. We hope that eventually you will be able to solve most of the problems. Good luck!

Total 59 marks.

1 Implicit Differentiation

In previous assignments we have differentiated functions of the explicit form $y = f(x)$, where y is written as a function of x. However, some functions are defined *implicitly* by an equation satisfied by y and x but where y is not the subject. The equation of a circle $x^2 + y^2 = r^2$ gives a familiar example of an implicit function.

You can use the standard rules of differentiation to find the derivative of implicit functions. More examples can be found in these Khan Academy videos: video 1, video 2, video 3.

Example 1. An implicit function is given $x^2 + y^2 = 4$. Find the derivative y' of y with respect to x, in terms of x and y.

Solution. Differentiating both sides with respect to x, we obtain

$(x^2)' + (y^2)' = 4'$

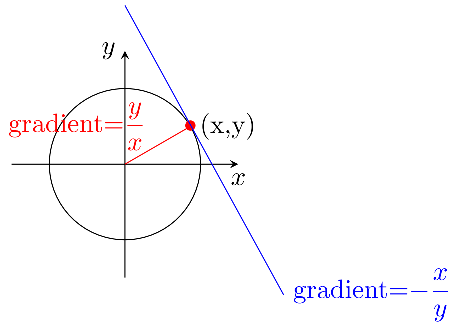
Note that the chain rule must be used for y^2: $(y^2)' = 2y \times y'$:

$2x + 2yy' = 0.$

Rearranging:

$$y' = -\frac{x}{y} \tag{1}$$

If we recognise the previous curve as a circle, then the result makes sense as the tangent[1] (with gradient y') is perpendicular to the radius (with gradient $\frac{y}{x}$).



One may argue that the above procedure can be avoided by simply solving the implicit function equation to obtain $y = \sqrt{4 - x^2}$, and then finding the derivative using the chain rule:

$$y' = \frac{(4 - x^2)'}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}},$$

which is the same as Eq. (1). However, the method for implicit differentiation we just studied obviates the need for equation solving, which may be challenging or even impossible. Furthermore, the solution $y = \sqrt{4 - x^2}$ implies a positive y, whereas the result (1) applies for all y's — positive or negative.

Problem 1 (4 marks). An implicit function is given by

a) $x^4 y - \cos(x + y) = 0$;

b) $8x^2y^2 - 7x^6 + \tan y = 5y - 7$.

Find the derivative y'_x in terms of x and y.

The next two questions deal with conic sections. If you haven't studied them at school, you can find information on the ellipse and hyperbola on this Khan Academy page.

Problem 2 (3 marks). Find the equations of the tangents to the ellipse

$$2x^2 + 3y^2 = 14$$

at the two points where $x = 1$. Find the coordinates of their point of intersection. Illustrate your result graphically.

Problem 3 (3 marks). Find the equations of the tangents to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (c, d). Show that none of these tangents pass through the origin. Choose a specific set of parameters $\{a, b, c, d\}$ and illustrate your result graphically.

[1] Refer to Y12 Mathematics Assignment 5 if you need to refresh your memory about tangents and normals.

2 Connected Rates of Change

Many practical situations involve several related variables. For example as the radius of a sphere changes, so do the surface area and the volume, or as the temperature of a gas changes, so do the volume and the pressure. In these situations the chain rule can be used to find the rates of change of each of the related variables.

Example 2. The volume of a sphere is increasing a a rate of $\frac{dV}{dt} = 20\ \text{cm}^3\text{s}^{-1}$. Find the rate $\frac{dr}{dt}$ at which the radius is increasing when the radius is $r = 6$ cm.

Solution. Differentiating V with respect to t using the chain rule yields

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$. Notice that this is the same expression as the surface area as the sphere — this is not a coincidence, try to explain why this is so. We have

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

Substituting the given values, we find

$$\frac{dr}{dt} \approx 0.0442\ \text{cm s}^{-1}.$$

More examples can be found in these Khan Academy videos: video 1, video 2, video 3.

Problem 4 (PAT 2006) (2 marks). The volume of a spherical balloon increases by 20 cm^3 every second. What is the rate of growth of the radius when the surface area of the balloon is 800 cm^2?

Problem 5 (PAT 2016) (4 marks). A cylinder of dough is squashed such that its height h decreases linearly with time t as $h(t) = h_0 - \alpha t$ for $t < h_0/\alpha$. Assume that the volume V of the dough remains constant, and it retains a cylindrical shape. Find an expression for the rate of change of the radius of the cylinder as a function of time and the parameters h_0, α, and V. Does the rate of change increase or decrease with time?

3 Derivative of the inverse function

Example 3. Find the derivative of $y = \ln(x)$.

Solution. Let us rewrite $y = \ln(x)$ as $x = e^y$. We can think of the latter expression as an equation of an implicit function $y(x)$. Differentiating both sides of this equation by x gives

$$1 = \frac{\mathrm{d}e^y}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Since $\dfrac{\mathrm{d}e^y}{\mathrm{d}y} = e^y$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^y}.$$

Since $e^y = x$, we find

$$\ln'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}.$$

Problem 6 (4 marks).

Show that

a) $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$;

b) $\arccos'(x) = -\dfrac{1}{\sqrt{1-x^2}}$.

Let us generalize this argument. Suppose the function $y = f(x)$ has a derivative on some interval $a < x < b$ and is invertible with $g = f^{-1}$ (see Y12 Mathematics Assignment 8 for the definition of an inverse function). Suppose we know the derivative $g'(y)$ of the function $g(y)$. Can we use it to find the derivative of $f(x)$?

Starting with the relationship $x = g(y)$ and differentiating with respect to x gives that

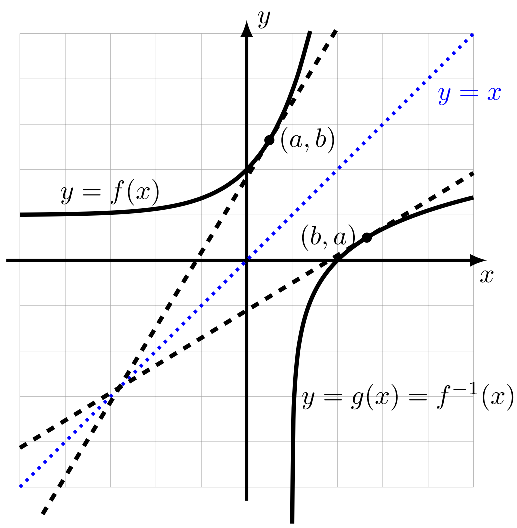
$$g'(y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

or

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{g'(y)}. \tag{2}$$

Please convince yourself that the results of Example 3 and Problem 6 are consistent with the general expression above.

To visualize the result (2), note that the graph of the inverse function $y = g(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$. Under this reflection, the tangent to $y = f(x)$ at (a, b) is reflected onto the tangent to $y = g(x)$ at (b, a) and the gradients of these two tangents are inverse of one another.



Problem 7 (6 marks). Find the derivatives of the functions $y = f(x)$ that are inverse to the following functions $x = g(y)$:

a) $x = \cosh y$; b) $x = \tanh y$; c) $x = \dfrac{e^y}{1+e^{2y}}$ for $y > 0$.

In the examples so far we were able to express our answers in terms of x. To that end, we had to solve the equation $x = g(y)$ to find the explicit form of the function $y = f(x)$, and then substitute this into Eq. (2). But sometimes this equation is difficult or even impossible to solve. In this case, like for implicit functions discussed in the previous section, we may need to keep the dependence on y in our answer for $f'(x)$.

Example 4.

a) Show that the function $g : \mathbb{R} \to \mathbb{R}$ defined as $x = g(y) = 2y + \sin y$ is invertible, and find the domain of the inverse.

b) If $f = g^{-1}$, find the equation to the tangent to the curve $y = f(x)$ at the point where $x = 2\pi$.

c) Is it possible to find a general expression for $f'(x)$?

Solution a) We have $f'(y) = 2 + \cos y$. As $-1 \le \cos y \le 1$ always, we see that $g'(y) > 0$. This implies that $g(y)$ is strictly increasing, and hence one-to-one and invertible on the entire set $\mathbb{R}$.

We can note that $f(x) \to \infty$ for $y \to \infty$ and $f(y) \to -\infty$ for $y \to -\infty$ so the range of g is $\mathbb{R}$. This implies that the domain of $f = g^{-1}$ is also $\mathbb{R}$.

b) We note that $g(\pi) = 2\pi + \sin\pi = 2\pi$. Using the relationship for the derivative of the inverse function f, we have

$$f'(2\pi) = \frac{1}{g'(\pi)} = \frac{1}{2+\cos\pi} = 1.$$

Since $g(\pi) = 2\pi$, we have that $f(2\pi) = \pi$ and hence the tangent passes through $(2\pi, \pi)$. The gradient of the tangent is 1, and we thus have that the tangent is $y = x - \pi$.

c) In general we have

$$f'(x) = \frac{1}{g'(y)} = \frac{1}{2+\cos y}$$

where y is the unique number satisfying the equation $2y + \sin y = x$. However, there is no easy way to get y in terms of x.

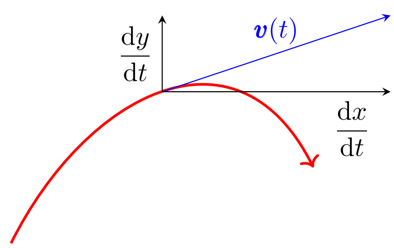
Problem 8 (2 marks). Show that $x = g(y) = y^3 + 3y$ has an inverse. If the inverse function is denoted by f, find the equation of the normal to the curve $y = f(x)$ at the point where $x = 4$.

4 Derivatives of a parametric function

Sometimes both y and x are given in terms of some third variable, known as the *parameter*. For example, $x(t)$ and $y(t)$ are the coordinates of a particle dependent on time t. Then $y(t)$ can be differentiated with respect to time using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$y' = \frac{\dot{y}}{\dot{x}},$$

where in the last line $\dot{y}$ and $\dot{x}$ represent differentiation with respect to time. Thinking physically, this last expression makes sense, if we note that the velocity vector $\boldsymbol{v}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$ points in the direction of the tangent to the path. The ratio $\frac{\dot{y}(t)}{\dot{x}(t)}$ gives the gradient of the velocity vector.



Problem 9 (3 marks).

Find the derivative $y'(x)$ (assume $a > 0, b > 0$), if:

a) $x = \sqrt[3]{1 - \sqrt{t}}, \quad y = \sqrt{1 - \sqrt[3]{t}}$.
b) $x = \sin^2 t, \quad y = \cos^2 t$.
c) $x = a \cos t, \quad y = b \sin t$.

Problem 10 (3 marks). For the function given parametrically by:

$$x = at^2;$$
$$y = 2at :$$

a) Sketch the curve in the (x, y) plane.

b) Find the equation of the normal to the curve at $t = t_0$.

c) Find the values of t_0 for which this normal passes through $(5a, 2a)$.

5 Functions of multiple variables

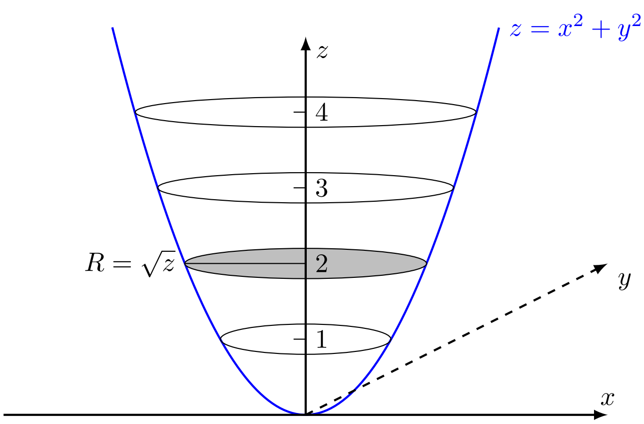
At this point in your studies you are very familiar with functions of a single variable, and know that $y = f(x)$ can be represented as a graph. It is also possible to define a function of two or more variables. A two-variable function $z = u(x, y)$ can be represented as a surface of height z, which varies dependent on x and y.

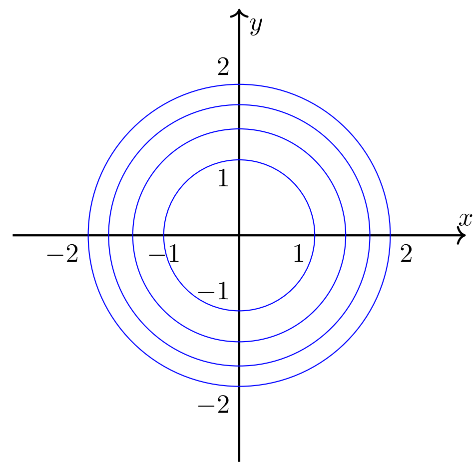
The usual method of representing a two-variable function graphically is the *contour plot*, which consists of lines connecting points (x, y) corresponding to the same z — akin to lines showing elevation on topographic maps. The values of z, for which contours are drawn, are typically chosen at equal distances (i.e. to represent an arithmetic progression).

Example 5. Sketch the surface

$$z = x^2 + y^2$$

Solution The contour $x^2 + y^2 = z$, which for $k > 0$ is a circle with radius $\sqrt{z}$. The curve, sketched below, is a paraboloid.





Problem 11 (8 marks). Sketch the following surfaces, and describe the contour lines:

a) $z = 2x + y$;

b) $z = e^{-x^2+y^2}$

c) $z = xy$;

d) $z = \dfrac{1}{x^2 + y^2 + 1}$.

6 Partial derivatives

6.1 First-order partial derivatives

Suppose that $u(x, y)$ is a real valued function and let x increase by a small amount h while y remains unchanged in value. The average rate of increase of u with respect to x is given by

$$\frac{u(x+h, y) - u(x, y)}{h}.$$

The limit as $h \to 0$, if it exists, is called the *partial derivative with respect to* x at the point (x, y) and is written $\dfrac{\partial u(x, y)}{\partial x}$. We can define a similar partial derivative by letting y vary and keeping x constant:

$$\lim_{h \to 0} \frac{u(x, y+y) - u(x, y)}{h} = \frac{\partial u(x, y)}{\partial y}. \tag{3}$$

In order to avoid a surfeit of symbol, one can often write $\dfrac{\partial u}{\partial y}$ if it is clear that this is being evaluated at the point (x, y).

The rules of ordinary differentiation (chain rule, product rule, quotient rule etc) can easily by extended to partial differentiation, as the only thing one needs to remember is that any variables aside from the one you are differentiating by need to be treated as if they were a constant.

Example 6. Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ for the functions

a) $u = x^3 y^2$;

b) $u = \ln(x^2 + y^2)$;

c) $u = x^y$.

Solution.

a) To find $\dfrac{\partial u}{\partial x}$ we differentiate the x^3 term with respect to x and leave the factor of y^2 unchanged, so $\dfrac{\partial u}{\partial x} = 3x^2 y^2$. Analogously we have $\dfrac{\partial u}{\partial y} = 2x^3 y$.

b) In both cases we use the chain rule to differentiate $\dfrac{\partial u}{\partial x} = \dfrac{2x}{x^2 + y^2}$ and $\dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2 + y^2}$.

c) In this case if y is treated as a constant then the function is simply a polynomial in x, and we have $\dfrac{\partial u}{\partial x} = yx^{y-1}$. If, however, we treat x as a constant, then the function is an exponential in y and to differentiate we write it as $u = \left(e^{\ln x}\right)^y = e^{y \ln x}$ which differentiates to $\dfrac{\partial u}{\partial y} = \ln x e^{y \ln x} = \ln x x^y$.

Problem 12 (3 marks). Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ in the following cases:

a) $u = x^5 y^2$;

b) $u = \dfrac{x^2}{y}$;

c) $u = \arctan \dfrac{x}{y}$.

Problem 13 (2 marks). Let $u(x,t) = \phi(x+ct) + \psi(x-ct)$, where $\phi(\cdot)$ and $\psi(\cdot)$ are arbitrary functions of a single argument. Verify that $u(x,t)$ satisfies the differential equation known as the *wave equation*[2]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

6.2 Second-order partial derivatives

Once you have found a partial derivative, with respect either to x or to y, then it is possible to form second, or higher, derivatives by computing the partial derivative of the result. The following notation is commonly used:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial^2 u}{\partial x^2};\\
\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) &= \frac{\partial^2 u}{\partial y \partial x};\\
\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial^2 u}{\partial x \partial y};\\
\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) &= \frac{\partial^2 u}{\partial y^2}.
\end{aligned}$$

Example 7. Find the four second-order partial derivatives for $u = x^2 \ln y$.

Solution. We have $\dfrac{\partial u}{\partial x} = 2x \ln y$. The partial derivatives of this function are

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 2\ln y;\\
\frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{2x}{y}.
\end{aligned}$$

On the other hand, $\dfrac{\partial u}{\partial y} = \dfrac{x^2}{y}$ and then

$$\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{2x}{y};\\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{x^2}{y^2}.
\end{aligned}$$

[2] This function $u(x,t)$ is known as the *D'Alembert solution* to the wave equation.

We can see here that the two mixed partial derivatives are both equal regardless of whether we differentiate first with respect to x and then with respect to y, or the other way around, i.e. $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$. This result is known as *Clairaut's theorem.*

Let us prove it. In spirit of Eq. (3), we write

$$\begin{aligned}\frac{\partial^2 u(x,y)}{\partial x \partial y} &= \frac{\partial}{\partial x}\frac{\partial u(x,y)}{\partial y} \\ &= \lim_{h_x \to 0} \frac{\dfrac{\partial u(x+h_x,y)}{\partial y} - \dfrac{\partial u(x,y)}{\partial y}}{h_x} \\ &= \lim_{h_x \to 0, h_y \to 0} \frac{\dfrac{u(x+h_x,y+h_y) - u(x+h_x,y)}{h_y} - \dfrac{u(x,y+h_y) - u(x,y)}{h_y}}{h_x}.\end{aligned}$$

If we change the order of x and y, the above expression changes to

$$\frac{\partial^2 u(x,y)}{\partial y \partial x} = \lim_{h_x \to 0, h_y \to 0} \frac{\dfrac{u(x+h_x,y+h_y) - u(x,y+h_y)}{h_x} - \dfrac{u(x+h_x,y) - u(x,y)}{h_x}}{h_y}.$$

We can see by inspection that the two expressions are the same.

6.3 Approximations and errors

In Y12, we learned about Taylor series[3]. The first-order Taylor approximation of a one-variable function $f(x)$ is $f(x + \Delta x) = f(x) + f'(x)\Delta x$. Let us generalize it to multiple variables. If $\phi(x,y)$ is a differentiable function and Δx and Δy, are small, then

$$\phi(x+\Delta x, y+\Delta y) - \phi(x,y) = \frac{\partial \phi}{\partial x}(x,y)\Delta x + \frac{\partial \phi}{\partial y}(x,y)\Delta y \tag{4}$$

To see this, let the positive x-axis point east, the positive y-axis point north.

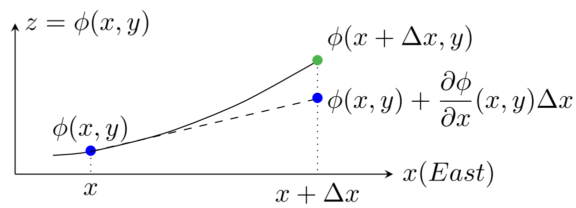
Imagine walking due east in a straight line from (x,y) to $(x+\Delta x, y)$. On this line the value of y is constant, so by the very definition of the partial derivative, we have, for small Δx that

$$\frac{\phi(x+\Delta x,y) - \phi(x,y)}{\Delta x} \approx \frac{\partial \phi}{\partial x}(x,y),$$

which rearranges to

$$\phi(x+\Delta x, y) \approx \phi(x,y) + \frac{\partial \phi}{\partial x}(x,y)\Delta x. \tag{5}$$

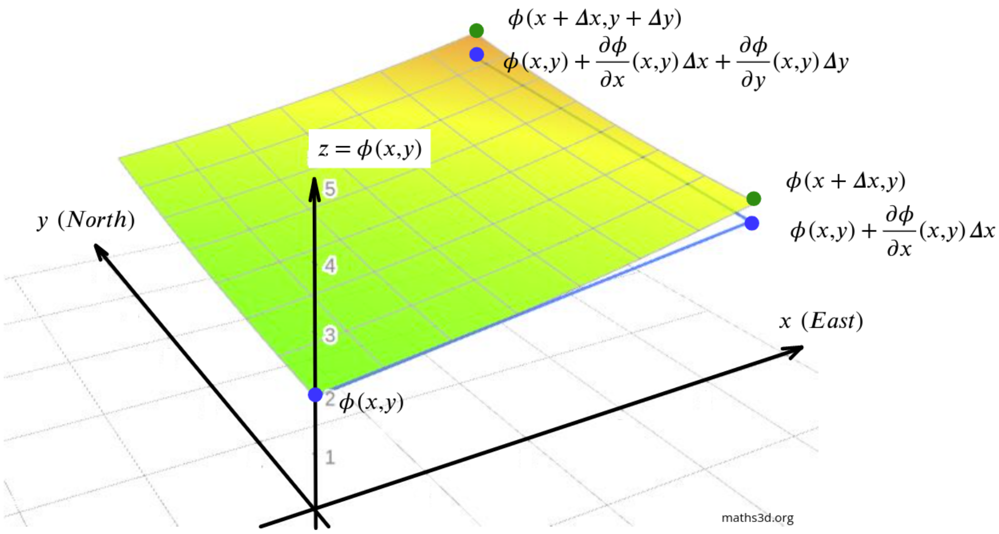
This is the usual approximation of the rate of increase of a function by the gradient of its tangent.



[3] If you need a reminder, you can watch this Khan academy video.

We can now consider walking due north from $(x + \Delta x, y)$ to $(x + \Delta x, y + \Delta y)$. By identical reasoning we have that

$$\phi(x + \Delta x, y + \Delta y) - \phi(x + \Delta x, y) \approx \frac{\partial \phi}{\partial y}(x + \Delta x, y)\Delta y. \tag{6}$$



Notice the partial derivative is computed at $(x + \Delta x, y)$ rather than the original point (x, y) because the second part of the journey starts from where the first part of the journey finished. But if Δx is small then $\frac{\partial \phi}{\partial y}(x + \Delta x, y) \approx \frac{\partial \phi}{\partial y}(x, y)\Delta y$.

Combining Eqs. (5) and (6), and taking the above approximation, we obtain Eq. (4).

Problem 14 (4 marks).

a) For $\phi = x^m y^n$ show that

$$\frac{\Delta \phi}{\phi} \approx \left(m\frac{\Delta x}{x} + n\frac{\Delta y}{y} \right). \tag{7}$$

b) Find an approximation for $\frac{\sqrt[3]{1003}}{\sqrt{102}}$. Do not use a calculator.

Suppose that, as a physicist, you are performing an experiment, in which you measure the variables x and y, and use your measurements to compute some function $\phi(x, y)$. Let the uncertainty of measuring x be Δx (i.e. x is known to within $\pm \Delta x$) and y with the uncertainty Δy. Equation (4) can then be used to estimate the uncertainty of ϕ.

Problem 15 (4 marks).

a) A train is moving at a speed (60 ± 1) km/h. A passenger is walking down the train isle at a speed (5 ± 1) km/h in the direction opposite to the train's movement. What is the uncertainty of the passenger's speed with respect to the ground?

b) A car is moving from rest with a constant acceleration (5 ± 0.1) m/s^2 for (5 ± 0.2) seconds. What is the uncertainty of the distance that the car has travelled?

c) A voltage (300 ± 1) V is applied to a resistor (600 ± 2) Ohm. What is the uncertainty of the current?

We hope you did not answer "zero" in any of the above questions! A measurement result can deviate from the true value to both positive and negative sides. When calculating the uncertainly of ϕ, you need to consider the worst case scenario, such that the signs of errors in x and y combine to give the highest deviation of ϕ.

Let us consider two specific cases to obtain the error propagation rules you may be already familiar with.

- Let $\phi = x \pm y$. Then Eq. (4) becomes $\Delta\phi = \Delta x \pm \Delta y$. This is formulated as the familiar rule that when you add or subtract two quantities, their *absolute* uncertainties must be added to obtain the absolute uncertainty of the result.
- Let $\phi = xy$ or $\phi = x/y$. Then, using Eq. (7), we find $\dfrac{\Delta\phi}{\phi} = \dfrac{\Delta x}{x} \pm \dfrac{\Delta y}{y}$. To put it in words, when you multiply or divide two quantities, their *relative* uncertainties must be added to obtain the relative uncertainty of the result.

6.4 Conservative forces

In Y12 you learned the concept of conservative force. The work W_{AB} performed by a conservative force moving a point object from point A to point B depends only on the location of these points, but not on the path taken. For conservative forces, one can define potential energy: a function $V(\vec{r})$ such that

$$W_{AB} = -(V_B - V_A),$$

where V_A and V_B are the potential energy values at the points A and B, respectively. We have seen that the weight and spring force are conservative.

The new message we would like to convey in this assignment is as follows. The components of a conservative force vector are equal to the partial derivatives of the potential energy by the respective coordinates[4]:

$$\vec{F}(\vec{r}) = \begin{pmatrix} F_x \\ F_y \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial V(\vec{r})}{\partial x} \\ \dfrac{\partial V(\vec{r})}{\partial y} \end{pmatrix}. \tag{8}$$

In other words, a conservative force is the negative gradient of its potential energy.

To see this, let us consider an element $\Delta\vec{r} = (\Delta x, \Delta y)$ of the path from A to B, which is sufficiently short to be approximated as straight. The corresponding work equals

$$W = -\Delta V = \vec{F} \cdot \Delta\vec{r} = F_x \Delta x + F_y \Delta_y,$$

[4] A vector of the form (8) — whose components are partial derivatives of some position-dependent scalar by the respective coordinates — is called the *gradient* of that scalar. Gradients are important, so they are assigned a special symbol, $\vec{\nabla}$ (read "nabla"):

$$\begin{pmatrix} \partial V(\vec{r})/\partial x \\ \partial V(\vec{r})/\partial y \end{pmatrix} \equiv \vec{\nabla} V(\vec{r}).$$

Note that previously we encountered the term "gradient" in the meaning of "derivative" or "slope". The gradient defined here is the extension of this term to multiple dimensions. For example, if $z(x, y)$ is the height of a hill, the vector $\vec{\nabla} z(x, y)$ shows the direction of the steepest accent up this hill and the slope steepness.

where ΔV is the change of the potential energies between the ending and starting points of the path. Comparing this with Eq. (4), we obtain the result (8).

This result is straightforwardly generalized to three dimensions. For example, in the case of gravitational field near the Earth surface, we have $V = mgz$. The components of the force are then equal to $F_x = -\partial V/\partial x = 0$, $F_y = -\partial V/\partial y = 0$, and $F_z = -\partial V/\partial z = -mg$, as expected.

Example 8. The potential energy of an object in three dimensions is given by $V = -K/r$, where K is a constant and r is the absolute value of its radius vector. Find the corresponding force vector.

Solution. We write $r = \sqrt{x^2 + y^2 + z^2}$. Differentiating the potential energy, we find

$$\vec{F} = -\begin{pmatrix} \partial V/\partial x \\ \partial V/\partial y \\ \partial V/\partial z \end{pmatrix} = \begin{pmatrix} -Kx/(x^2+y^2+z^2)^{3/2} \\ -Ky/(x^2+y^2+z^2)^{3/2} \\ -Kz/(x^2+y^2+z^2)^{3/2} \end{pmatrix} = -\frac{K}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This can be rewritten simply as $\vec{F} = -K\vec{r}/r^3$ or $\vec{F} = -K\hat{r}/r^2$, where $\hat{r} = \vec{r}/r$ is the vector of magnitude 1 in the direction of $\vec{r}$. In other words, the force's direction is towards the origin, and its magnitude is proportional to the square of the distance to the origin. Examples of such force are electrostatic attraction of a point charge or gravitational attraction of a point mass.

Given a force vector's components, can we tell whether it is conservative? For two dimensions, there is a simple recipe. From Eq. (8), we can see that $\frac{\partial F_x}{\partial y} = \frac{\partial^2 V}{\partial y \partial x}$ and $\frac{\partial F_y}{\partial x} = \frac{\partial^2 V}{\partial x \partial y}$. In other words, for a conservative force, the derivative of F_x with respect to y must be equal to the derivative of F_y with respect to x.

Problem 16 (4 marks). Which of the forces below are conservative? If the force is conservative, find the corresponding potential energy.

a) $F_x = y, F_y = -x$;

b) $F_x = y, F_y = x$;

c) $F_x = x, F_y = y, F_z = z$.