Comprehensive Oxford Mathematics and Physics Online School (COMPOS)

Year 13

Physics Assignment 01

Rotational Mechanics

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Due 14th September 2025

Total 50 marks.

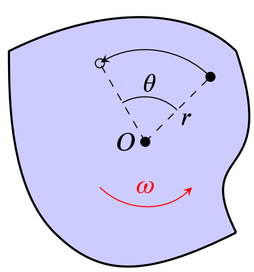
In this assignment we will consider the *angular* motion of a point particle about an axis, or the motion of an extended body (which can be modelled as being made of many point particles) about an axis.

Some aspects of rotational motion are really non-intuitive. We recommend watching this Lecture by Walter Lewin to get a taste of angular motion.

1 Rotational Kinematics

Rotational kinematics have been mostly covered in this Y12 assignment on circular motion. Please study the first section of that assignment to refresh your knowledge of angular displacement, angular velocity and angular acceleration. You will recall the analogy between linear and rotational motion. The key quantities correspond as follows:

Linear Motion	**Rotational Motion**
Displacement (x)	Angular displacement (θ)
Velocity (v)	Angular velocity (ω)
Acceleration (a)	Angular acceleration (α)



These parallels allow us to apply intuition and problem-solving techniques from linear motion to rotational motion. Assuming constant angular acceleration, the kinematic equations of rotational motion are very similar to the linear kinematic equations (SUVAT equations):

Rotational Motion	Linear Motion
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
$\theta = \frac{1}{2}(\omega + \omega_0)$	$x = \frac{1}{2}(v + v_0)t$

Example 1. A spinning disk starts from rest and accelerates uniformly at an angular acceleration of $\alpha = 4.0\,\mathrm{rad/s^2}$. How many revolutions will it make in the first 5.0 s, and what is its final rotational frequency?

Solution. We use the angular kinematic equations for constant angular acceleration. The final angular velocity is $\omega = \omega_0 + \alpha t = 20.0\,\mathrm{rad/s}$, which corresponds to the frequency $f = \omega/2\pi = 3.18$ Hz. The angular displacement is given by $\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 50.0\,\mathrm{rad}$, i.e. $\theta/2\pi = 7.96$ revolutions.

Problem 1 (2 marks). A jet engine turbine uniformly accelerated from 10 to 100 revolutions per second in 2 seconds. How many revolutions has it made in the process?

2 Rotational kinetic energy and moment of inertia

Consider a rigid body with some mass distribution rotating about some axis with angular velocity ω. Let us calculate the total kinetic energy of this rotation, assuming that the object is made up of many small masses m_i. Each such mass, located at distance r_i away from the axis of rotation, is moving with speed $v_i = \omega r_i$ and its kinetic energy is $T_i = \frac{1}{2}m_i\omega_i^2 r_i^2$. The total KE is then $T = \sum_i \frac{1}{2}m_i\omega^2 r_i^2 = \frac{1}{2}\omega^2 \sum_i(m_i r_i^2)$.

For a continuous object the sum can be replaced by an integral:

$$I = \sum_i (m_i r_i^2) \rightarrow I = \int r^2 \mathrm{d}m$$

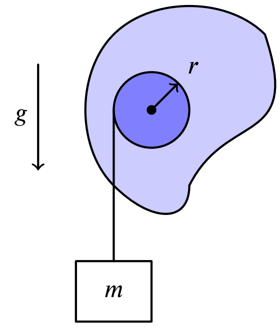
Here, r is the perpendicular distance from the axis of rotation to an infinitesimal[1] mass element dm. This integral depends on the shape and mass distribution of the object and is referred to as the *moment of inertia* of the object. The kinetic energy then becomes

$$T = \frac{1}{2}I\omega^2. \tag{1}$$

Comparing this with the KE for translational motion, $T = \frac{1}{2}mv^2$, and recalling that the angular velocity ω is the analogue of velocity v of translational motion, we conclude that the moment of inertia is the rotational analogue of mass m. It is the physical quantity that characterizes the rotational inertia of a body about a given axis.

Problem 2 (2 marks). A rigid body is mounted on a horizontal axis passing through its centre of mass. A light drum of radius r, rigidly attached to the body, is mounted on the same axis. A weight of mass m is suspended from the free end of a string wound around the drum. The weight, some time after having been released from rest, has descended a distance h and reached a velocity v. Find the moment of inertia of the body.

[1] Infinitely small.



Example 2. Find the moment of inertia of a thin ring of mass M and radius R rotating around the axis passing through its centre orthogonally to its plane.

Solution. The orbit of each point of the ring has radius R. Hence the formula for the moment of inertia can be applied trivially:

$$I = \int r^2 \mathrm{d}m = R^2 \int \mathrm{d}m = R^2 M.$$

Example 3. Find the moment of inertia of a uniform thin disk of mass M and radius R rotating around the axis passing through its centre orthogonally to its plane.

Solution. Now the situation is less trivial, because the radii r associated with different elements of the disk are different. Let us divide the disk into a set of small rings of radius r and infinitesimal width dr. The area density of the disk is $M/\pi R^2$, hence the mass of each infinitesimal ring is its area $2\pi r\mathrm{d}r$ times this density: $\mathrm{d}m = \frac{2M\pi r\mathrm{d}r}{\pi R^2} = \frac{2Mr\mathrm{d}r}{R^2}$. Hence

$$I = \int r^2 \mathrm{d}m = \int_0^R \frac{2Mr^3}{R^2}\mathrm{d}r = \left[\frac{2Mr^4}{4R^2}\right]_0^R = \frac{MR^2}{2}.$$

We see that the moment of inertia of a disk is less than that of a ring of the same mass and same radius. This is not surprising: while in a ring all points move with the same linear velocity ωR, points in the body of the disk move with smaller velocities, resulting in a smaller total kinetic energy (assuming that the disk and the ring are rotating at the same ω).

It is also easy to see that the thickness of the disk does not matter: the same formula will apply if the disk is replaced by a long cylinder.

Problem 3 (2 marks). Show that the moment of inertia of a uniform thin rod of mass m and radius R rotating around the axis passing through its centre orthogonally to it is $ML^2/12$.

Problem 4 (3 marks). Show that the moment of inertia of a uniform thin disk of mass M and radius R relative to the axis, which contains the diameter of the disk, is $MR^2/4$.

Hint: if you are struggling with the integral, try introducing a variable θ such that $r/R = \sin\theta$.

The table below shows the moments of inertia for common solid bodies. Try deriving them independently.

Object	Axis of Rotation	Moment of Inertia
Point mass	Distance r from mass	mr^2
Thin rod	Through centre, perpendicular to rod	$\frac{1}{12}ML^2$
Solid cylinder	Cylinder axis	$\frac{1}{2}MR^2$
Solid thin disk	Diameter	$\frac{1}{4}MR^2$
Solid sphere	Diameter	$\frac{2}{5}MR^2$
Hollow sphere	Diameter	$\frac{2}{3}MR^2$
Rectangular plate $a \times b$	Through centre, perpendicular to plane	$\frac{1}{12}M(a^2+b^2)$

If I_{cm} is the moment of inertia of a body about an axis through its centre of mass, then the moment of inertia about a parallel axis a distance D away is:

$$I = I_{\text{cm}} + MD^2$$

This result is known as the *parallel axis theorem (Huygens–Steiner theorem)* and is frequently used when calculating rotational inertia about different axes. Its proof can be found in many online sources, for example in this video by Jason Zhu.

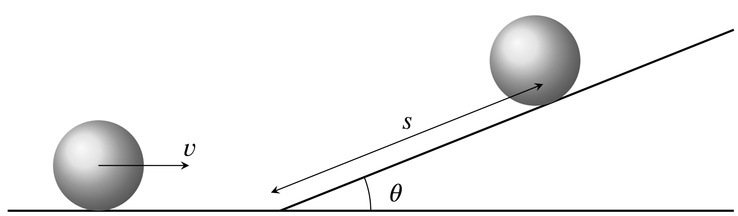
Problem 5 (2 marks). Find the moment of inertia of a thin uniform rod of about the axis passing through its end orthogonally to it

a) from the first principles;

b) using the parallel axis theorem.

Check that your results are consistent.

Example 4. A solid uniform sphere (mass m, radius R) is rolling without slipping on a horizontal surface with centre-of-mass speed v. It then encounters a rough incline of angle θ and rolls up the slope (no energy is lost to frictional dissipation; static friction only enforces rolling without slipping). What distance s measured along the slope will the sphere travel before coming momentarily to rest?



Solution. Because there is no dissipative friction, mechanical energy is conserved. The initial kinetic energy (translational + rotational) is converted into gravitational potential energy at the maximum height.

For a rigid body rolling without slipping,

$$\omega = \frac{v}{R},$$

where v is the speed of the centre of mass and ω the angular speed. The moment of inertia of a solid sphere about its centre is

$$I = \frac{2}{5}mR^2.$$

Total kinetic energy initially:

$$K_{\text{initial}} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v^2}{R^2}\right) = \frac{1}{2}mv^2 + \frac{1}{5}mv^2 = \frac{7}{10}mv^2.$$

Let the ball climb a distance s along the slope. The vertical rise is

$$h = s\sin\theta.$$

At the turning point the sphere's kinetic energy is zero and its gain in gravitational potential energy is

$$\Delta V = mgh = mgs\sin\theta.$$

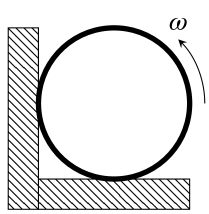
Applying energy conservation, we find

$$s = \frac{7}{10}\frac{v^2}{g\,\sin\theta.}$$

Equivalently the maximum vertical rise is $h_{\text{max}} = \frac{7}{10}\frac{v^2}{g}$. The factor $7/10$ reflects the partition of energy between translation and rotation for a solid sphere. If the object were sliding without rotation (or a point mass), the result would be $h = v^2/(2g)$ and $s = v^2/(2g\sin\theta)$.

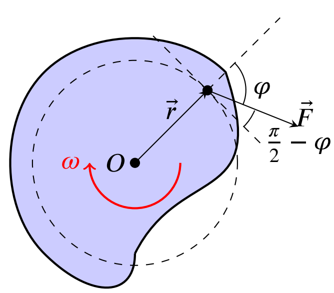
Problem 6 (2 marks). A solid ball, a hula-hoop and a hollow ball (shell) are rolling at constant speed when they encounter a rough inclined plane. Find the ratio of the distances travelled by each object up the slope before momentarily coming to rest. Does this distance depend on the radii or masses?

Problem 7* (3 marks). A thin-walled cylinder is spinning with angular velocity ω. It is placed next to a wall so that there is friction between the cylinder and both the wall and floor. The coefficient of friction is μ. Find the number of complete rotations of the cylinder before stopping.



3 Torque

Let us now derive the analogue of Newton's Second Law for rotation.



Consider an external force acting in such a way that it always makes the same angle φ with the radius-vector $\vec{r}$ of the application point of the force. In other words, the force "rotates" with the object. To calculate the work done by such a force, we recall that the path travelled by the point where the force is applied is always perpendicular to the radius, i.e. at angle $\frac{\pi}{2} - \varphi$ to the force:

$$W = Fs\cos\left(\frac{\pi}{2} - \varphi\right) = Fs\sin\varphi.$$

The distance travelled by the force is $s = \theta r$, where θ is the angular displacement, so the work done is

$$W = Fr\sin\varphi \times \theta = \tau\theta,$$

where $\tau = Fr\sin\varphi$ is the *moment of the force*, or *torque*,[2] familiar to you from GCSE physics classes and the Y10 assignment on simple mechanisms. Drawing again the analogy with the work in translational motion, $W = Fs$, and recalling that angular displacement θ is the analogue of linear displacement s, we conclude that the torque is the rotational equivalent of linear force.

3.1 Newton's Second Law for rotational motion.

Suppose we apply a torque τ to an object with moment of inertia I rotating freely around some axis. What will be its angular acceleration α?

Suppose the object is initially rotating around a fixed axis with angular velocity ω_0. We apply the torque for time t, after which its angular velocity becomes ω_1. From the Work-Energy principle we know that the change in the kinetic energy equals the work done by the torque:

$$T_1 - T_0 = W.$$

$$\frac{1}{2}I\omega_1^2 - \frac{1}{2}I\omega_0^2 = \tau\theta.$$

Dividing both sides by $I/2$ and making ω_1^2 the subject, we obtain

$$\omega_1^2 = \omega_0^2 + 2\frac{\tau}{I}\theta,$$

which is the same form as the equation of kinematics (see the second table in Sec. 1) $\omega_1^2 = \omega_0^2 + 2\alpha\theta$. Comparing the two equations we can see that

$$\tau = I\alpha \tag{2}$$

where τ is the net torque, I is the moment of inertia, and α is the angular acceleration. We have obtained Newton's Second Law for rotational motion, whose equivalent in translational dynamics is the familiar $F = ma$.

To summarise the above discussion, here is a table of related quantities for rotational and translational motions.

[2] The moment of force is sometimes defined as $\tau = Fr_\perp$, where $r_\perp = F\sin\varphi$ is the component of the radius that is perpendicular to the force.

Comparison with Linear Dynamics

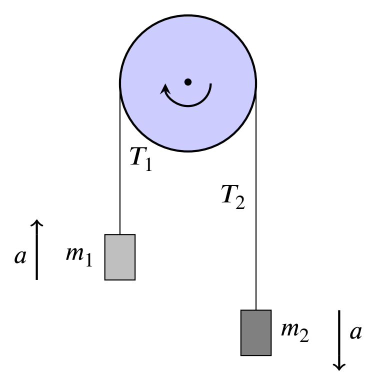
	Linear Quantity	**Rotational Analogue**
	Force F	Torque $\tau = rF \sin\theta$
	Mass m	Moment of inertia $I = \int r^2\, dm$
	Acceleration a	Angular acceleration $\alpha = \frac{d\omega}{dt}$
	Momentum mv	Angular momentum $I\omega$
Newton's Second Law	$F = ma$	$\tau = I\alpha$
Work	$W = Fd$	$W = \tau\theta$
Kinetic energy	$T = \frac{1}{2}mv^2$	$T = \frac{1}{2}I\omega^2$
Power	$P = Fv$	$P = \tau\omega$

Problem 8 (2 marks). A flywheel of mass m in the shape of a ring of radius R with light spokes is initially spun at angular velocity ω and subsequently stops and due to friction. Find the torque of the friction force if:

a) the flywheel stopped after time t;

b) the flywheel did N complete revolutions before stopping;

Example 5. Two masses m_1 and m_2 are connected by a light, inextensible string that runs over a pulley of mass M and radius R. The pulley is a solid uniform cylinder. There is no slipping between the string and the pulley. Find the linear acceleration a of the masses. Assume $m_2 > m_1$.



Solution. Choose signs and coordinates as follows:

- Let the upward direction for m_1 be positive. Since $m_2 > m_1$, m_1 accelerates upward with magnitude a.
- For m_2 take downward as positive (so it accelerates downward with magnitude a).
- The pulley rotates with angular acceleration α. No slip gives $\alpha = a/R$.

Let T_1 and T_2 be the tensions in the two segments of string. Note that tensions are different — otherwise there would be no torque on the pulley and it won't turn. The equations for Newton's Second Law for the two masses are then

$$T_1 - m_1 g = m_1 a;$$

$$m_2 g - T_2 = m_2 a.$$

Let us now write the rotational equivalent of Newton's Second Law for the pulley. Tensions on the two sides exert a net torque $(T_2 - T_1)R$ (positive when $T_2 > T_1$, corresponding to m_2 going down). The angular acceleration is $\alpha = a/R$. Hence

$$(T_2 - T_1)R = I\frac{a}{R}.$$

Substituting T_1 and T_2 from the first two equations into the third one, we have

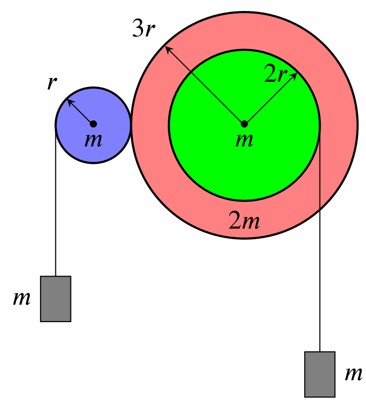
$$(m_2 - m_1)g - \frac{I}{R^2}a = (m_1 + m_2)a.$$

For the given solid uniform cylindrical pulley $I = \frac{1}{2}MR^2$, so $I/R^2 = \frac{1}{2}M$. Substituting,

$$a = \frac{(m_2 - m_1)\,g}{m_1 + m_2 + \frac{1}{2}M}.$$

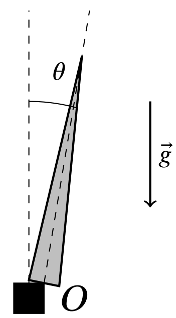
Problem 9 (3 marks). Solve Example 5 using the work-energy principle.

Problem 10 (4 marks). In the figure below, the green and red pulleys are glued together while the blue and red pulleys do not slide with respect to each other due to strong friction. All three pulleys are solid cylinders with parameters as shown. Find the acceleration of the system.



Problem 11 (MIPT, 4 marks). A tall and thin uniform pole of length l is falling, with its bottom pivoting on the ground. Find the angular acceleration α and the angular velocity ω in terms of the instantaneous angle θ between the pole and the vertical.

Problem 12* (MIPT, 4 marks). A tall tree of height $H = 28$ m and mass $m = 1000$ kg has the shape of a circular cone. The tree is cut down and starts to fall. Find the kinetic energy of the tree as it lifts off the stump. Assume that the tree is pivoting on the stump with the centre of the circular cross-section touching the edge of the stump. The tree lifts off when the component of the reaction force along the trunk is zero.



In the problems above, the axis of rotation is well-defined. But what if there is no such axis? Situations like this are common: for example, a football rolling on a field or a spaceship changing its orientation in space.

In cases like this, you can still apply the techniques we developed, but you need to choose the axis — an imaginary line around which the object rotates. This choice is not trivial. The arguments we made above, like all mechanics studied so far, are valid in inertial reference frames. Hence the axis chosen needs to represent such a frame: it must be stationary in space or move with a constant velocity.

Fortunately, it is allowed for such an axis to be *instantaneous*. For example, if a football is rolling without sliding, its bottom point is instantaneously stationary: it is not moving with respect to the ground. So we can apply the torque rules around this axis[3].

But what if there is no obvious axis that is instantaneously stationary? In this case, we can also use the line passing through the *centre of mass* of the rotating object, even if it is not an inertial reference frame. The proof of this statement requires the use of the vector cross product and can be found in the Appendix.

Example 6. A solid sphere of mass M and radius R is released from rest and rolls without slipping down an incline angled at θ to the horizontal. Find the linear acceleration a of the centre of the sphere.

Solution. We will solve the problem using three methods.

Method 1. The horizontal line passing through the point where the sphere touches the incline (marked red in the diagram) is the instantaneous rotation axis. Let us write the rotational equivalent of Newton's Second Law with respect to this axis.

There are three forces acting on the sphere: gravity Mg, normal reaction N and friction f. The latter two happen to be applied at the touching point, so their torque is zero. The gravity force is applied at the sphere's centre, distance R away from the axis, so its torque is $\tau = MgR\sin\theta$.

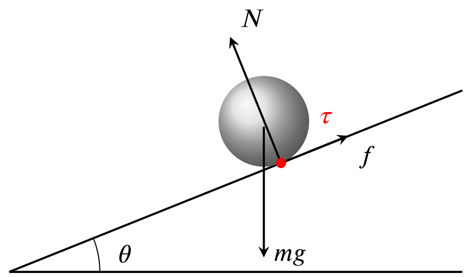
The moment of inertia of a solid sphere around its centre is $\frac{2}{5}MR^2$. To find its moment of inertia around the instantaneous rotation axis, we apply the parallel axis theorem: $I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$.

Finally, notice that the angular acceleration α around the instantaneous rotation axis equals $\alpha = a/R$, where a is the linear acceleration of its centre that we aim to find.

[3] There is a subtlety: even though the bottom point of the ball is not moving, it is still experiencing *normal* acceleration. So, strictly speaking, it is not an inertial reference frame. However, normal force is perpendicular to the trajectory and hence does no work, so our derivation of the rotational equivalent of Newton's Second Law is not affected. Alternatively, one can say that the instantaneous axis is not the bottom point of the wheel, but the point on the road directly underneath it — which is stationary.

Putting it all together, we have

$$\tau = I\alpha \;\Rightarrow\; MgR\sin\theta = \frac{7}{5}MR^2\frac{a}{R} \;\Rightarrow\; a = \frac{5}{7}g\sin\theta.$$



Method 2. Let us instead choose the sphere's centre as the axis. Now the torques of gravity and normal reaction forces vanish, but the friction has non-zero torque $\tau = fR$. This torque causes the acceleration

$$\tau = I\alpha = \frac{2}{5}MR^2 \times \frac{a}{R},$$

so

$$f = \frac{2}{5}Ma.$$

Now we have two unknowns (a and f) abut only one equation. An additional equation comes from Newton's 2nd Law for the ball's *translational motion* (which holds in spite of the rotation!). Along the incline:

$$Mg\sin\theta - f = Ma$$

Solving for a, we find

$$a = \frac{5}{7}g\sin\theta$$

We see that the acceleration is less than $g\sin\theta$ we would have seen if the sphere were replaced by a frictionless solid block. To understand the nature of this discrepancy (and also to dispel doubts if you are uncomfortable with the previous arguments), let us solve the same problem using energy considerations.

Method 3. Suppose the sphere rolled for time t with acceleration a. This means that it gained a velocity $v = at$ and travelled a distance $s = \frac{at^2}{2}$. The potential energy due to gravity has decreased by $Mgs\sin\theta$. The kinetic energy due to translational motion is $K_t = \frac{1}{2}mv^2$. In addition, there is kinetic energy due to the sphere's rotation about its centre of mass: $K_r = \frac{1}{2}\times\frac{2}{5}MR^2\times\omega^2$, where the angular velocity $\omega = \frac{v}{R}$ because the sphere rolls without sliding. The total kinetic energy is hence

$$K = K_t + K_r = \frac{1}{2}Mv^2 + \frac{1}{5}MR^2\frac{v^2}{R^2} = \frac{7}{10}Mv^2.$$

Applying energy conservation, we obtain

$$Mgs\sin\theta = \frac{7}{10}Mv^2.$$

Using the SUVAT relation $v^2 = v_0^2 + 2as$ with $v_0 = 0$, we find

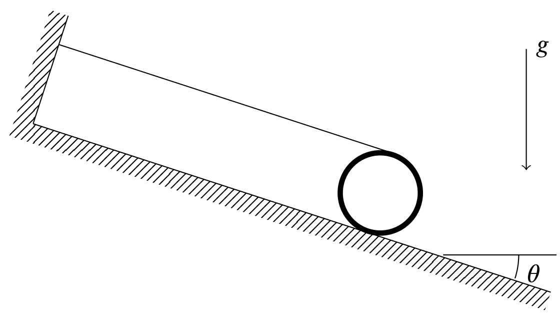
$$a = \frac{v^2}{2s} = \frac{5}{7}g\sin\theta.$$

We see that the acceleration is reduced in comparison to frictionless sliding because because some energy goes into rotation. You will recall a demonstration to this effect in the Lecture by Walter Lewin you watched in the beginning of the assignment.

Problem 13 (3 marks). A solid ball or radius R and mass m is spun to angular velocity ω around a horizontal axis and placed on a rough horizontal surface. The ball first accelerates, and then starts to roll. Find the final speed of the ball once it is rolling without slipping.

Problem 14 (2 marks). Find the maximum angle of the incline such that a hollow ball (e.g. a basketball) will roll down without slipping. The friction coefficient is μ.

Problem 15 (Savchenko, 3 marks). A thin-walled cylinder has a thread wound around it. The other end of the thread fixed so that, as the cylinder slips down an inclined plane, the thread remains parallel to the plane. What speed does the cylinder acquire after its axis has travelled a distance l from rest? The inclination angle of the plane is θ and the coefficient of friction between the plane and the cylinder is μ.



4 Angular Momentum

By analogy with the momentum $m\vec{v}$ in translational motion, we can define *angular momentum* for rotational motion:

$$L = I\omega. \tag{3}$$

Differentiating both sides of (3) with respect to time, and using the rotational equivalent (2) of Newton's Second Law, we obtain

$$\frac{d\vec{L}}{dt} = I\dot{\vec{\omega}} = I\vec{\alpha} = \vec{\tau}, \tag{4}$$

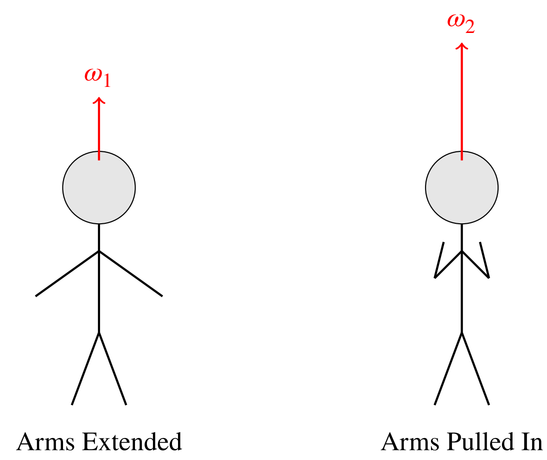
so the change in the angular momentum equals the impulse of the torque. If no net torque is acting on the system, or if we observe the system for a short enough time to make the torque impulse insignificant, the angular momentum is conserved[4]:

$$\vec{L}_{\text{initial}} = \vec{L}_{\text{final}}. \tag{5}$$

This conservation law is a cornerstone of mechanics.

Example 7. This example refers to this video by OpenStax. A figure skater is spinning with her arms outstretched. She has an initial moment of inertia of $I_1 = 4.0\,\text{kg}\cdot\text{m}^2$ and an initial angular velocity of $\omega_1 = 2.0\,\text{rad/s}$. When she pulls her arms inward, her moment of inertia decreases to $I_2 = 2.0\,\text{kg}\cdot\text{m}^2$. What is her final angular velocity ω_2?

[4] You may wish to refresh your memory of the Y12 assignment on momentum to see the analogy in the argument.



Solution. Since there is no external torque acting on the skater, angular momentum is conserved:

$$L_1 = L_2 \Rightarrow I_1\omega_1 = I_2\omega_2.$$

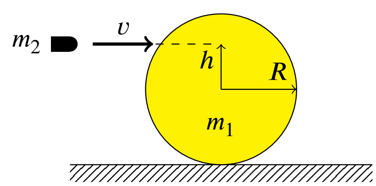
Solving for ω_2:

$$\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{4.0 \times 2.0}{2.0} = 4.0\,\text{rad/s}.$$

It is interesting to look at her initial and final rotational kinetic energies. These energies are different: $K_1 = \frac{1}{2}I_1\omega_1^2 = 8.0\,\text{J}$ and $K_2 = \frac{1}{2}I_2\omega_2^2 = 16.0\,\text{J}$. How is this possible when there is no net torque to do the work? The answer is that she does the work while pulling her arms in. More precisely, the work is done by the centripetal force acting on her arms while she is spinning.

Problem 16 (3 marks). Between 1992 and 2020, polar ice sheets (including Antarctica and Greenland) lost a combined 7,560 billion tonnes of ice. Estimate the resulting change in the duration of the day.

Problem 17 (Savchenko, 3 marks). A bullet of mass m_2, moving with speed v, hits a wooden cylinder of mass m_1, as shown in the diagram, and sticks therein. Assuming $m_2 \ll m_1$, determine the linear and angular velocities of the cylinder immediately after the impact.



Problem 18 (Savchenko, 3 marks). A hoop of radius R spinning in the vertical plane falls vertically onto a horizontal plane and bounces off it with speed v at an angle of 30°, no longer spinning. What was the angular velocity of the hoop before the impact?

Appendix. Why centre of mass can be used as axis in torque problems.

We will be using the cross product and its properties, which will be studied in detail in a future assignment on 3D geometry. Let us suppose the angular acceleration is $\vec{\alpha}$. Then the linear acceleration of a given point of the object with the radius-vector r with respect to the centre of mass is a vector sum of the centre-of-mass-acceleration $\vec{a}$ and the acceleration $\alpha \times \vec{r}$ due to the angular acceleration. If $\alpha \neq 0$, there will be some point A, with the radius-vector $\vec{r}_0$, in which these two accelerations cancel each other. This point is the instantaneous axis of rotation, and we can write

$$\vec{\alpha} = \frac{\vec{\tau}_A}{I + Mr_0^2}, \tag{6}$$

where I is the moment of inertia around the centre of mass (so $I + Mr_0^2$ is the moment of inertia around A) and $\vec{\tau}_A$ is the torque around A.

Let us relate $\vec{\tau}_A$ and the torque $\vec{\tau}_{\text{CM}}$ around the centre of mass. Suppose the object is acted upon by several forces $\vec{F}_i$ applied at points with radius-vectors $\vec{r}_i$ with respect to the centre of mass. Then the radius-vector of each such point with respect to point A is $\vec{r}_i - \vec{r}_0$, so

$$\vec{\tau}_{\text{CM}} = \sum_i \vec{r}_i \times \vec{F}_i$$

and

$$\vec{\tau}_A = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_i = \sum_i \vec{r}_i \times \vec{F}_i - \vec{r}_0 \times \sum_i \vec{F}_i = \vec{\tau}_{\text{CM}} - \vec{r}_0 (M\vec{a}),$$

where $\vec{a} = \sum_i \vec{F}_i / M$ is the linear acceleration of the centre of mass.

We now notice that the difference between the accelerations of$A$ and the centre of mass is $\vec{a}_A - \vec{a} = \vec{\alpha} \times \vec{r}_0$. Because point A is not accelerating, we have $\vec{a} = \vec{a} - \vec{a}_A = -\vec{\alpha} \times \vec{r}_0$. Hence

$$\vec{\tau}_A = \vec{\tau}_{\text{CM}} + M\vec{r}_0 \times [\vec{\alpha} \times \vec{r}_0].$$

Using properties of the cross product,

$$\vec{r}_0 \times [\vec{\alpha} \times \vec{r}_0] = \vec{\alpha} \cdot (\vec{r}_0 \cdot \vec{r}_0) - \vec{r}_0 \cdot (\vec{\alpha} \cdot \vec{r}_0) = \vec{\alpha} r_0^2.$$

The last equality is valid because $\vec{\alpha} \perp \vec{r}_0$, so $\vec{r}_0 \cdot (\vec{\alpha} \cdot \vec{r}_0) = 0$. Putting it all together into Eq. (6), we find

$$\vec{\alpha} = \frac{\vec{\tau}_{\text{CM}} + M\vec{\alpha} r_0^2}{I + Mr_0^2} \;\Rightarrow\; \vec{\alpha} I + \vec{\alpha} M r_0^2 = \vec{\tau}_{\text{CM}} + M\vec{\alpha} r_0^2 \;\Rightarrow\; \vec{\alpha} = \frac{\vec{\tau}_{\text{CM}}}{I},$$

meaning that we can use the rotational equivalent of Newton's Second Law to rotation around the centre of mass.